Firm heterogeneity, financial frictions and ambiguity [Preliminary and incomplete]

Lorenzo Carbonari* Filippo Maurici†

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Abstract

*This paper studies the effects of ambiguity (*Knightian uncertainty*) on business cycle and inequality in an economy with heterogeneous agents. We introduce ambiguity averse entrepreneurs in a growth model with financial frictions and market-wide source of ambiguous information. Entrepreneurs use a worst-case criterion to formulate expectation on the total factor productivity and are heterogeneous in terms of assets and productivity. We find that ambiguity: i) increases the productivity threshold to access the market, ii) does not alter the relative consumption gap between producing and non-producing entrepreneurs and iii) increases the consumption gap between entrepreneurs and workers. We also find that, in the long-run, an economy featuring ambiguity accumulates more assets and produces more. This is the consequence of the entrepreneurs' hedging strategy which also entails lower wages.*

Keywords: economics growth, ambiguity, collateral constraints, heterogeneous agents, transition dynamics.
JEL codes: E22, D81, D84, G14, O11.

*Università di Roma "Tor Vergata", DEF and CEIS. E-mail address: lorenzo.carbonari@uniroma2.it.
†Università di Roma "Tor Vergata". Corresponding author. E-mail address: filippo.maurici@uniroma2.it.

1 Introduction

A traditional question in macroeconomics is what are the main drivers of output fluctuations. In this regard, a recent branch of the literature has investigated the implications of shocks to agent's confidence. A distinctive feature of these models is that agents are assumed averse to ambiguity, a situation in which the probabilities of some relevant events are not necessarily objectively known. As a consequence, ambiguity averse agents take their decision using a worst case probability drawn from a set of multiple beliefs. In this paper we investigate the interplay between ambiguity and financial frictions in business cycle model with heterogeneous agents. To do this we introduce ambiguity, as modelled in Epstein and Schneider (2007) and Ilut and Schneider (2014), into Buera and Moll (2015).

The economy is composed by entrepreneurs and workers, living infinitely in discrete time. Each period is divided into two stages. In the first stage, agents make agreements according to pessimistic expectation on the total factor productivity (TFP). Agreements cannot be updated once ambiguity disappears. In the second stage, the TFP is realized and production takes place. Workers are homogeneous while entrepreneurs are heterogeneous with respect to assets and productivity. Since, at each period, not all entrepreneurs could find it convenient to produce we will distinguish between producing (*active*) and non-producing (*non-active*) entrepreneurs.

The contribution of the paper is twofold. First, we show that, in the short-run, aggregate production goes in the same direction of confidence shocks. A negative shock on confidence increases the minimum level of productivity required to enter the market, thus determining a reduction in the number of active entrepreneurs. Business cycles arise because entrepreneurs hedge against the worst-case scenario, by lowering the level of production. Moreover, the expected decrease in TFP implies a reduction of the wage rate, which is determined in the same sub-period at which the confidence shock occurs. The combination of the two effects turns out to be benign for those entrepreneurs who actually produce. Profits goes up and so do capital accumulation. As time goes by, if the confidence shock is persistent the economy converges to a long-run equilibrium characterized by higher capital and output and lower wages.

Our second contribution is to show that shocks to entrepreneurs' confidence affect consumption inequality in favour of entrepreneurs. Pessimistic expectations induce capital accumulation, which, in turn, implies higher future consumption for entrepreneurs. The consumption ratio between active and non-active producers does not change with respect to the case with no uncertainty, i.e. the Buera and Moll (2015)'s economy. However, both groups consume more in absolute terms. On the other hand, workers consume less, triggering an increase in consumption inequality between workers and entrepreneurs.

Related literature We build upon Buera and Moll (2015), which studies the aggregate implications of a credit crunch with different specifications of heterogeneity. Their analysis reveals that the presence of heterogeneity plays a crucial role in understanding the sources of business cycles. A similar set-up can also be found in Banerjee and Moll (2010) and Itskhoki and Moll (2019). Banerjee and Moll (2010) introduces the idea of distinct types of capital misallocation, intensive and extensive, disentangling the quantity of capital from the quality of its usage, attributing persistent inefficiencies to binding financial constraints and heterogeneity. Itskhoki and Moll (2019) sets the same model in continuous time to study optimal development policies and dynamic taxation.

Recent papers describe the right tail of the wealth distribution as a Pareto (see e.g. Buera and Moll (2015), Gabaix et al. (2016), Achdou et al. (2017) and Itskhoki and Moll (2019)). Along this line, we model the heterogeneity among entrepreneurs as a productivity shock with a Pareto distribution. In the aggregate, this will determine a productivity threshold such that only entrepreneurs with a sufficiently high level of productivity will actually produce.

We describe ambiguity aversion according to the axiomatization presented in Epstein and Schneider (2003). We model ambiguity following Epstein and Schneider (2007) and Ilut and Schneider (2014). Epstein and Schneider (2007) analyzes a setting where agents confidence changes as they learn. Similarly, Ilut and Schneider (2014) describes a model featuring confidence shocks and shows how changes in ambiguity creates business cycles.

The multi-stage structure of the economy is borrowed by Angeletos and La'O (2013). This structure introduces another layer of friction in the economy. In their case, islands cannot communicate and agents receive signal from the matched trader. Before the trade happens both players must produce. What they find is that random fluctuations in the signals induce matched agents to change production according to the random shock.

Finally, the fact that ambiguity induces capital accumulation for precautionary motives is reminiscent of Aiyagari (1994).

Outline The paper is structured as follows. In Section 2 we lay out the structure of the economy while in Section 3 we derive the optimal choices and characterize the competitive equilibrium. In Section 4 we present a numerical analysis of the models. Section 5 is a conclusion. The proofs are in the Appendix.

2 Environment

We consider a small-open economy set in discrete time with an infinite horizon. There are a production sector and an external financial sector. In the production sector firms operate in a competitive market producing a single final good, combining labor and capital. While workers are homogeneous and hand-to-mouth, entrepreneurs are heterogeneous in their productivity (z) and asset holdings (a).

Entrepreneurs are affected by the *Knightian uncertainty* and are ambiguity averse. We assume that there is market-wide information about the set of *perceived* distributions ($\mathcal{P}_t$). Ambiguity is represented by all the possible realization $P \in \mathcal{P}_t$. Each period is dived into two stages.

Sequence of the events We assume that each period is divided into two subperiods (stages). At the beginning of the first stage each entrepreneur observe the realization of the idiosyncratic productivity shock z while there is a market-wide ambiguity about the realization of the TFP, A_t. During the first stage, production and consumption choices are taken, while in the second stage they actually occur. At the beginning of the second stage, A_t is revealed. An important assumption is that agents cannot change their decisions on consumption and production during the second stage.

Financial market There is a risk-free internationally traded asset, whose interest rate is denoted by r^*. Financial markets are incomplete, in that active entrepreneurs face a collateral constraint given by:

$$k_t \leq \lambda a_t. \tag{1}$$

with $\lambda > 1$.

Ambiguity Ambiguity affects the economy through two channels: the TFP (A_t) and the future idiosyncratic productivity, $\{z_{t+q+1}\}_{q\in\mathbb{N}}$. The perceived possible realizations are described by the set of distributions $\mathcal{P}_t$, with $\mathcal{P}_t = \mathcal{A}_t \times \{\mathcal{Z}_{t+q+1}\}_{q\in\mathbb{N}}$, where $\mathcal{A}_t$ and $\{\mathcal{Z}_{t+q+1}\}_{q\in\mathbb{N}}$ are, respectively, the sets of perceived distributions from which the TFP and the future idiosyncratic productivity can be drawn. $\mathcal{P}_t$ is known market-wide.

The economy also has information about the minimum each ambiguous variable can attain. We denote the minimum outcome among each set as $\min \mathcal{A}_t = \underline{A}_t$ while $\min \mathcal{Z}_{t+q+1} = 1$, for all $q \in \mathbb{N}$. The keep the analysis easily tractable we assume that a Dirac-delta distribution $\mathcal{D}(\underline{A}_t, 1) \in \mathcal{P}_t$. Assuming this will allow to ignore any other element of $\mathcal{P}_t$. In fact, in the case of the Dirac-delta distribution the upper and the lower bounds coincide at $\underline{A}_t$, for the expected TFP, while for all the other elements of $\mathcal{P}_t$ the upper bound must be strictly higher than $\underline{A}_t$. This implies that ambiguity averse agents always will consider only $\mathcal{D}\left(\underline{A}_t, 1\right)$, i.e. the Dirac-delta characterizes the worst-case scenario.

Workers The representative worker's preferences are given by

$$U_w(c_{w,t}, \ell_t), \tag{2}$$

where $c_{w,t}$ is the individual consumption of a worker. Workers are hand-to-mouth so that, in each period, they consume all their wages accumulate no assets. As a consequence a worker's budget constraint is always binding

$$c_{w,t} = w_t \ell_t. \tag{3}$$

Entrepreneurs Entrepreneurs are heterogeneous in productivity (z) and assets holdings (a). They set consumption and production choices during the first stage. Entrepreneurs are pessimistic in the sense that their preferences are Epstein-Schneider, i.e.

$$U(\mathbf{c}) = \min_{P \in \mathcal{P}_t} \mathbb{E}_P \left[\sum_t \beta^t u(c_t) \right]. \tag{4}$$

where $\mathbf{c} = \{c_t, c_{t+1}, \dots\}$, for all t. To characterize the set of *perceived* distributions $\mathcal{P}_t$, we assume that the idiosyncratic productivity is drawn from a Pareto with parameter η_t, that can be allowed to change over time. The support of $F_t(z)$ is $[1, \infty)$ and its density function is $\eta_t z^{-\eta_t - 1}$.
To reconcile the sets of perceived and realized distributions it suffices to define $\mathcal{Z}_{t+q} = \{F \mid F \sim Pareto \text{ and } \eta \geq 1\}$. In fact, $\mathcal{D}(1) \in \mathcal{Z}_{t+q}$, corresponding to $\eta \to \infty$.

At each date t, the entrepreneur's production function is given by

$$A_t(z_t k_t)^\alpha n_t^{1-\alpha}, \tag{5}$$

while profits are

$$\pi_t(z_t) = A_t(z_t k_t)^\alpha n_t^{1-\alpha} - w_t n_t - r^* k_t. \tag{6}$$

An entrepreneur's wealth evolves according to

$$a_{t+1} = \pi_t(z_t) + (1 + r^*) a_t - c_t. \tag{7}$$

Notice that the dynamics of the entrepreneurial wealth is affected by the decisions on consumption and production taken in the first stage and by the exogenous realization of the TFP, which takes place in the second stage.

3 Optimal choices

In the first stage, workers choose their consumption and labor supply while entrepreneurs decide how much capital and labor use to maximize their profits. Expectations on TFP don't enter the workers' maximization problem. Moreover, since they are hand-to-mouth their decision is static. On the contrary, expectation on TFP affect the entrepreneurial behavior. Hence, they select the ambiguity minimizer in $\mathcal{P}_t$ and maximize (6) for any $P \in \mathcal{P}_t$. Let s_t^P denote the expectation of any s_t under the distribution P.

Stage 1 Workers choose labor, ℓ, and consumption, c_w, to maximize:

$$U_w(c_{w,t}, \ell_t) - \mu\left[c_{w,t} - w_t \ell_t\right]. \tag{8}$$

Optimality implies the standard condition:

$$\frac{\partial U_w / \partial \ell}{\partial U_w / \partial c} = w_t. \tag{9}$$

The entrepreneur is given by

$$\max_{n_t \geq 0, k_t \leq \lambda a_t} \pi_t^P(z_t) = A_t^P (z_t k_t)^\alpha n_t^{1-\alpha} - w_t n_t - r^* k_t. \tag{10}$$

Maximizing out the choice of labor and using (1) allows to rewrite y_t^P as a linear function of capital, for any $P \in \mathcal{P}_t$. At the equilibrium the constraint (1) is binding, so that $k_t = \lambda a_t$.

We can now prove the following.

Lemma 1
For any $P \in \mathcal{P}_t$, factor demands is linear in wealth and can be written as

$$k_t^P(z_t) = \lambda a_t \, \mathrm{I}(z_t \geq \underline{z}^P), \tag{11}$$

$$n_t^P(z_t) = \left[\frac{(1-\alpha)A_t^P}{w_t}\right]^{\frac{1}{\alpha}} z_t \lambda a_t \, \mathrm{I}(z_t \geq \underline{z}^P), \tag{12}$$

$$\pi_t^P(z_t) = \left[\frac{z_t}{\underline{z}^P} - 1\right] r^* \lambda a_t \, \mathrm{I}(z_t \geq \underline{z}^P), \tag{13}$$

where the threshold $\underline{z}^P$ is defined by the zero-profit condition:

$$\underline{z}^P = \max\left\{\frac{r^*}{\alpha}\left[\frac{w_t}{1-\alpha}\right]^{\frac{1-\alpha}{\alpha}} (A_t^P)^{-\frac{1}{\alpha}}, 1\right\}. \tag{14}$$

Condition (14) tells us that the cut-off level of z is constrained to be at least 1 because of the support of F_t. Such a condition can be binding depending from the starting conditions of the economy, however its long-run behavior is deterministic if no aggregate shock happens. Notice that if r^* is sufficiently low, *all* entrepreneurs will find it optimal to enter the market. For the sake of simplicity, we restrict our attention to a parameter space such that $\frac{r^*}{\alpha}\left[\frac{w}{1-\alpha}\right]^{\frac{1-\alpha}{\alpha}}(A_t^P)^{-\frac{1}{\alpha}} \geq 1$ for any $P \in \mathcal{P}_t$.[1]

Given the assumption of Epstein-Schneider preferences for the entrepreneurs, the expected profits maximization problem becomes:

$$\min_{P\in\mathcal{P}_t}\max_{n\geq 0,\, k\leq\lambda a} \pi^P(z). \tag{15}$$

We can now state the following.

Lemma 2

$\mathcal{D}(\underline{A}_t, 1)$ is the optimal distribution for hedging against ambiguity.

Since production and consumption decisions are taken before the realization of A_t, using (14) Lemma 2 implies

$$\underline{z} = \frac{r^*}{\alpha}\left[\frac{w_t}{1-\alpha}\right]^{\frac{1-\alpha}{\alpha}}\underline{A}^{-\frac{1}{\alpha}}. \tag{16}$$

Notice that the pessimistic expectation $\underline{A}_t$ affects the determination of the threshold, which cannot be updated *ex post*. It is clear that, under the ambiguity minimizer $\mathcal{D}(\underline{A}_t, 1)$, entrepreneurs choose consumption during the first stage *as if* the worst-case scenario is certain: this turns the stochastic maximization problem into a deterministic worst-case maximization.

An active entrepreneur consumes:

$$c(z_t) = (1-\beta)\left[(1+r^*)a_t + \pi^{\mathcal{D}}(z_t)\right], \tag{17}$$

while a non-producing entrepreneur consumes:

$$\tilde{c}(z_t) = (1-\beta)(1+r^*)a_t, \tag{18}$$

where the decision to produce depend on whether the realized z_t is below or above $\underline{z}$.

Rewrite the the expected law of motion of individual capital as

$$a_{+1}^{\mathcal{D}} = \begin{cases} \beta(1+r^*)a & \text{if } z \leq \underline{z}, \\ \beta\pi^{\mathcal{D}} + \beta(1+r^*)a & \text{if } z \geq \underline{z}. \end{cases} \tag{19}$$

It is now useful to define some aggregate equilibrium conditions, under the $\mathcal{D}(\underline{A}_t, 1)$ expectation. Denote as x the aggregate level of wealth, $\int a\, \mathrm{d}i$. The aggregate level of

[1] We consider the effects of $\underline{z} = 1$ in Appendix B.

capital and labor demand is

$$x_t \int_{\underline{z}} \lambda \, \mathrm{d}F = \lambda \underline{z}^{-\eta} x_t, \tag{20}$$

$$\ell_t = \left[\frac{(1-\alpha)\underline{A}}{w_t} \right]^{\frac{1}{\alpha}} x_t \int_{\underline{z}} z_t \lambda \, \mathrm{d}F. \tag{21}$$

The aggregate consumption for producing entrepreneurs is

$$C_P = (1-\beta) \left(1 + r^* + \frac{r^* \lambda}{\eta - 1} \right) \underline{z}^{-\eta} x, \tag{22}$$

and for non-producing entrepreneurs

$$C_U = (1-\beta)(1+r^*) \left(1 - \underline{z}^{-\eta} \right) x_t. \tag{23}$$

Define the average consumption for active and non-active entrepreneurs as

$$\overline{C}_P = (1-\beta) \left(1 + r^* + \frac{r^* \lambda}{\eta - 1} \right) x_t, \tag{24}$$

$$\overline{C}_U = (1-\beta)(1+r^*) x_t. \tag{25}$$

Finally, the aggregate expected production is

$$y_t^{\mathcal{D}} = \underline{A} \left[\int_{\underline{z}} z_t \lambda \, \mathrm{d}F \right]^{\alpha} x_t^{\alpha} \ell_t^{1-\alpha}. \tag{26}$$

We can now state the following.

Lemma 3

1. *Equilibrium expected aggregate output satisfies*

$$y_t^{\mathcal{D}} = \Theta(\underline{A}) x_t^{\gamma} \ell_t^{1-\gamma}, \tag{27}$$

where

$$\Theta(\underline{A}) = \left[\underline{A} \lambda^{\alpha} \Lambda^{\alpha \frac{1-\eta}{\eta}} \right]^{\frac{\eta}{\eta + \alpha(1-\eta)}},$$

$$\gamma = \frac{\alpha}{\eta + \alpha(1-\eta)} < 1.$$

2. *The productivity cut-off $\underline{z}$ is given by*

$$\underline{z}_t^{\eta} = \Lambda \frac{x_t}{y_t^{\mathcal{D}}}, \tag{28}$$

with $\Lambda = \frac{\eta \lambda}{\eta - 1} \frac{r^*}{\alpha}$.

3. *The aggregate cost of labor is*

$$w_t \ell_t = (1-\alpha) y_t^{\mathcal{D}}. \tag{29}$$

4. *The expected law of motion of the aggregate wealth is*

$$x_{t+1}^{\mathcal{D}} = \beta\alpha y_t^{\mathcal{D}} - \beta r^* \lambda \underline{z}_t^{-\eta} x_t + \beta(1+r^*)x_t. \tag{30}$$

Ambiguity affects production choices through the level of expected aggregate TFP, $\Theta(\underline{A}_t)$. This effect is mediated by both the parameter which captures the financial frictions, λ, and the skill-heterogeneity parameter, η_t. Moreover, the worse are the expectations (about the lower-bound $\underline{A}$) the higher is the productivity threshold $\underline{z}$.

Notice finally that the stage structure of the economy defines another friction layer. This is clear from the determination of $y_t^{\mathcal{D}}$, where the only possible changes affect the expected aggregate TFP but not the allocation choices in terms of wealth and labor.

Stage 2 In the second stage, the economy receives the information about the actual value of the TFP, A_t. As we know, however, agents decisions cannot be updated accordingly. Differences between the expected TFP and the realized one trigger business cycles fluctuations.

For a given realization of A_t, a competitive equilibrium, under the ambiguous expectation $\mathcal{D}(\underline{A}_t, 1)$, can be summarized as a time path for aggregate variables $\left\{c_{w,t}, \ell_t, x_{t+1}^{\mathcal{D}}, y_t^{\mathcal{D}}, \underline{z}\right\}_{t\in\mathbb{N}}$ and $\{w_t\}_{t\in\mathbb{N}}$, given an initial entrepreneurial wealth x_0.

Denoting as G_t the joint distribution of assets and productivity, we can now state the following.

Proposition 1
For any $A_t \geq \underline{A}_t$ and $z \geq \underline{z}$:

1. *the realized individual and aggregate profits are increasing in A_t, such that*

$$\pi(z_t) - \pi^{\mathcal{D}}(z_t) \propto (A_t - \underline{A}_t) \tag{31}$$

and

$$\int \pi(z_t) - \pi(z_t)^{\mathcal{D}} \mathrm{d}G = \left[\frac{A_t}{\underline{A}_t} - 1\right] y_t^{\mathcal{D}}; \tag{32}$$

2. *the realized law of motion of the aggregate wealth is*

$$x_{t+1} = \left[\frac{A_t}{\underline{A}_t} - 1 + \beta\alpha\right] y_t^{\mathcal{D}} - \beta r^* \lambda \underline{z}^{-\eta} x_t + \beta(1+r^*)x_t. \tag{33}$$

Pessimistic expectations lead entrepreneurs to accumulate assets for precautionary motives. Ambiguity implies an *ex ante* hedging process such that all the entrepreneurs who would not find it optimal to produce, once A_t is realized, prefer to stay out the market. This selection between active and non-active producers occurs through the productivity threshold $\underline{z}$. An important feature of the model is that entrepreneurs systematically underestimate the effective realization of the TFP. Equilibrium wages and consumption choices, which are determined before the realization of A_t, do not adjust *ex post*. Whenever $A_t > \underline{A}$, profits increase and wealth accumulation accelerates, fueled by lower wages and a lower consumption. The realized law of motion of wealth can be rewritten as

$$x_{t+1} = \left[\frac{A_t}{\underline{A}_t} - 1\right] y_t^{\mathcal{D}} + x_{t+1}^{\mathcal{D}}. \tag{34}$$

In this model, ambiguity aversion is not only a source of business cycle fluctuations but also of inequality in consumption. As for the entrepreneurs, consumption inequality arises from the alternation between productive and unproductive periods, depending on the realization of z_t. The larger is the accumulated wealth, the higher is the consumption level. As for the workers, since wages are determined during the first stage, according to pessimistic expectations, they are *de facto* excluded from the gains deriving from $A_t > \underline{A}_t$. For the same set of parameters, we compare our results with those of Buera and Moll (2015), who study an economy like ours with no uncertainty and ambiguity aversion. In our model, in which the ratio between active and non-active entrepreneurs is constant at the equilibrium, entrepreneurs consume more while workers consume less, i.e. ambiguity increases consumption inequality.

Discussion An important assumption we made is that a Dirac-delta distribution belongs to the set of perceived distributions. The assumption greatly simplify the calculations but it comes at a cost. The Dirac-delta allows to transpose an optimization where one of the choices is the distribution according to which one weighs the future, into the deterministic worst-case optimization, getting rid of the stochasticity both in the sense that the distribution is fixed and in the sense that the distribution is in itself composed by a point, then non-random.

The Dirac-delta distribution centered in the ambiguity lower bounds rules out any potential bizarre feature driven by the actual realization of the TFP. It operates as a sort of hedging distribution, since no entrepreneur can make losses if his expectation is set on the worst-case scenario. Having excluded the possibility to adjust *ex post*, allows entrepreneurs to gain further profits. Notice that, even in the absence of uncertainty and ambiguity aversion, the financial friction captured by (1) suffices to ensure positive profits to active entrepreneurs (see Itskhoki and Moll (2019)).

If the assumption about the Dirac-delta distribution were relaxed then the burden of the worst-case scenario would not be paid solely by the workers. On the other hand, the two-staged structure of the economy would require the introduction of contingent contracts. From a general viewpoint, the economy we described has some features in common with a typical Aiyagari economy (Aiyagari (1994)): agents are heterogeneous with respect to some characteristics and a form of uncertainty, risk or ambiguity, tackles the economy. The way uncertainty affects individual choices is via an increase in precautionary savings. This effect is preserved and amplified in our model. The precautionary motives arise not because of an idiosyncratic feature but rather due to the ambiguous TFP which affects the economy as a whole, through market-wide information. In this model, the consequences are radical since only those who will find it optimal to produce in the worst case scenario actually enter the market. To do so they use their wealth as collateral to acquire physical capital.

4 Quantitative analysis

We characterize the ambiguity as being the output the true TFP multiplied by a normal distribution with mean M and time-varying variance (σ_t^2): the dynamics of the variance reflects the learning process. The output is then constrained to be in the exogenous set $(m_t, A_t]$.

$$\underline{A} = \begin{cases} m & \text{if } A \cdot \mathcal{N}(M, \sigma) \leq m, \\ A \cdot \mathcal{N}(M, \sigma) & \text{if } m < A \cdot \mathcal{N}(M, \sigma) \leq A, \\ A & \text{if } A \cdot \mathcal{N}(M, \sigma) > A. \end{cases} \tag{35}$$

It is possible to characterize each model we have taken into account by the evolution of the variance. If $\sigma_t = 0$ identically, we have the model of Buera and Moll (2015). If $\sigma_t = \sigma > 0$ identically, we have our plain model. We use the formulation above to consider exogenous learning, as in Benhabib et al. (2015), that is a case where σ_t decreases over time. We call them complete and partial learning. The latter happens when $\sigma_t \to \sigma_{PL} < \sigma$, the former when $\sigma_t \to \sigma_{CL} = 0$. For the sake of exposition we define these processes as

$$\sigma_{PL+1} = \chi \sigma_{PL} + (1 - \chi) \frac{A_t}{\underline{A}_t}, \tag{36}$$

and

$$\sigma_{CL+1} = \chi \sigma_{CL}, \tag{37}$$

with σ_0 given and $\chi \in (0, 1)$. The process with partial learning is prevented from converging to 0 by the extra term $\frac{A_t}{\underline{A}_t}$, which takes into account the previous mistakes in prediction increasing uncertainty. On the other hand, the process with complete learning updates its variance correctly and converges to 0.

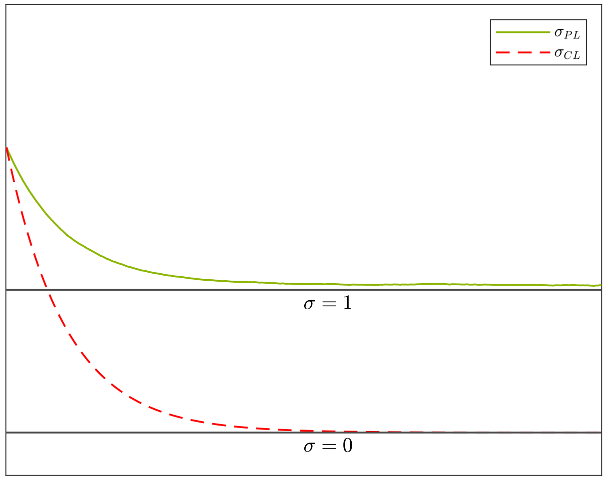

Figure 1: Dynamics of the variances with complete and partial learning.

Param.	Value	Description
λ	2	tightness of financial constraint
β	0.95	discount rate of entrepreneurs
α	0.33	capital share
η	2.05	inverse skill-heterogeneity
ψ	2	marginal disutility of work
φ	2	inverse Frisch elasticity
ρ	0.3	IRF memory
χ	0.99	learning rate
A	1	true TFP/ambiguity upper bound
m	0.9	ambiguity lower bound
M	1	ambiguity mean
x_0	10	initial aggregate assets
σ_0	2	initial variance
r^*	0.03	interest rate

Table 1: Parameter Values for Quantitative Exercise.

The ratios of aggregate consumption suggest that, once compared to the dynamics in Buera and Moll (2015), non-producing workers reduce their distance with respect to the producing ones while increasing it with respect to workers. This effect is mainly due to the fact that producing workers, even though they have access to the market and make higher profits, are still pushing down their consumption rules because of

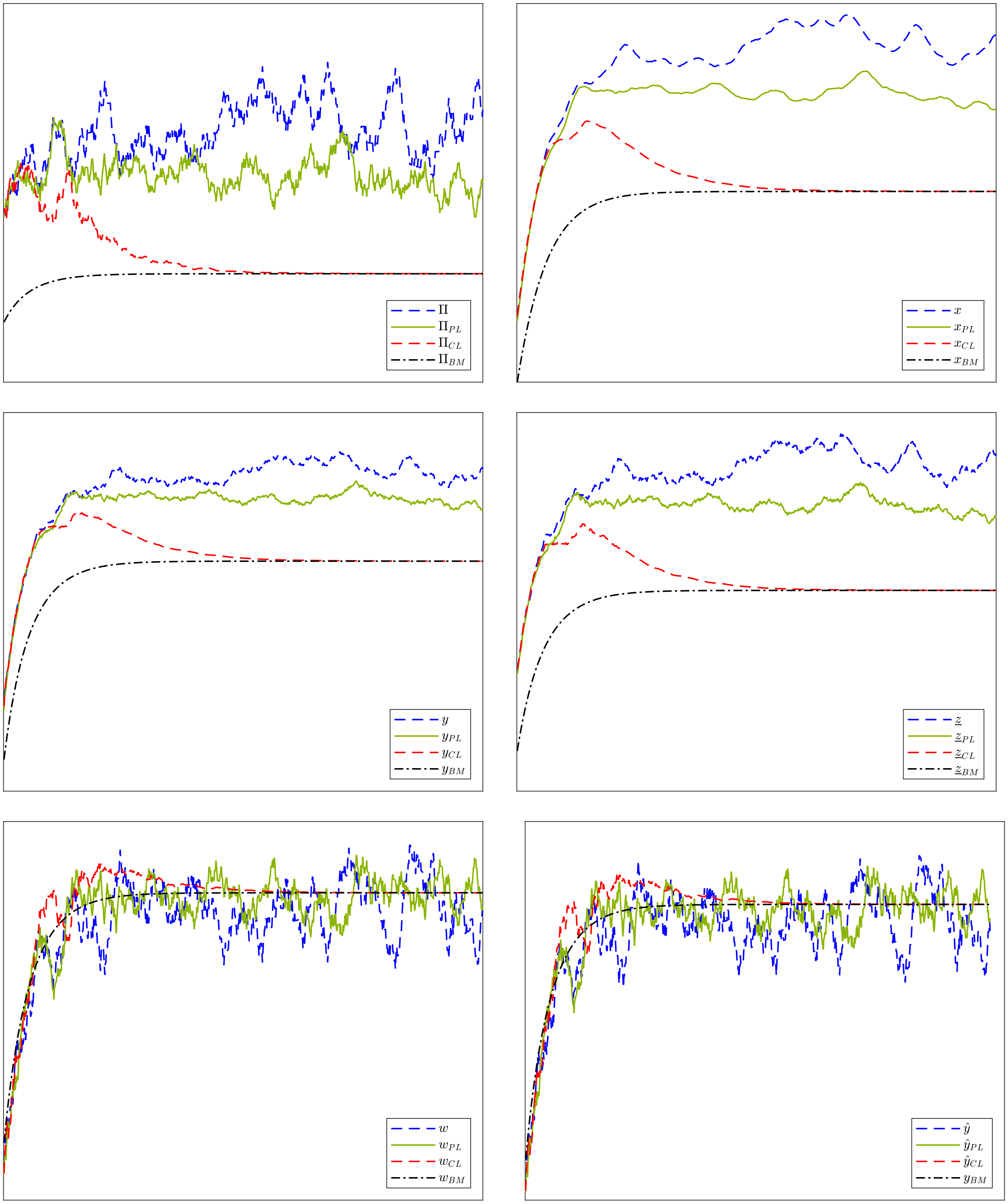

Figure 2: Dynamics of the model of Buera and Moll (2015), our plain model, our model with complete learning and our model with partial learning. All series are MA-smoothed.

ambiguity, the exact same effect pushes down wages and reduces workers' consumption. The reason why this does not happen in the case of non-producing entrepreneurs is because they do not face ambiguity precisely because they decide not to enter the market. These results suggest that, although ambiguity generates an unambiguous increase in $\underline{z}$, its effects must be distinguished among each group. On average, both workers and producing entrepreneurs consume less, but keep their ratios unchanged, that is they reduce consumption by the same proportion. there is at least one group, the non-producing entrepreneurs, that is better off with respect to the model without ambiguity, since they are able to accumulate more when they are certain to produce and consume this over-accumulation when they decide to stay out of the market.

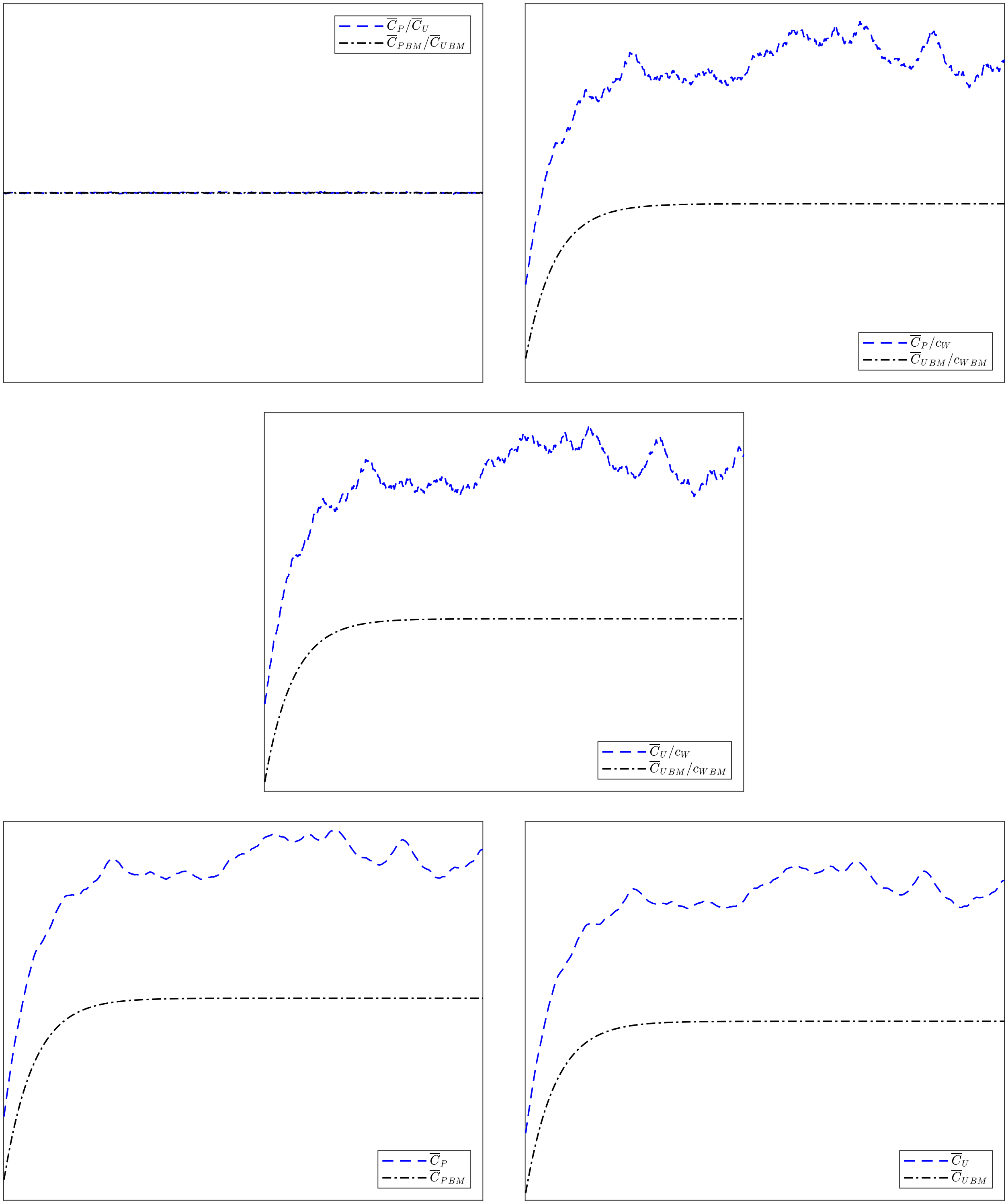

Figure 3: Dynamics of consumption by group of Buera and Moll (2015) and our plain model. All series are MA-smoothed.

Up to this point, it goes without saying that our model, not introducing ambiguity, collapses to that of Buera and Moll (2015). Thus, we determine the dynamics generated by an ambiguity shock in the TFP. Assuming that in the long-run ambiguity disappears ensures that both models converge to the same steady state so that impulse responses describe the short-run dynamics generated by the shock.

It is worth emphasizing that during the first stage, when nominal variables are set, it is impossible to disentangle, namely *to know*, whether the shock is a real TFP shock or an ambiguity shock. So the variables set in the first stage behave the same in the two models.

All the dynamics depends on the relation between the stickiness generated during the first stage and the impossibility to adjust market's choices in the second stage. This process keeps going in the subsequent periods carrying on the instability generated by the shock up to convergence back to the equilibrium. Following Buera and Moll (2015), we model the law of motion of the shocked process as

$$\underline{A}_{+1} = \rho \underline{A} + (1-\rho) A^{SS}, \tag{38}$$

with $\rho \in (0,1)$ and A^{SS} being the common steady state of the TFP.

As already stated, the overall dynamics depends on the first-stage choices. However, while in the original model what happens in the first stage carries on in the second, the same does not hold for our model, if expectation are not confirmed.

Starting with the common features, a negative TFP shock pushes down output. As a consequence, since the share of production given to labor is constant, wages shrink. Finally, the cut-off threshold increases due to a higher minimum level of idiosyncratic productivity to compensate the common TFP drop-off, this, as well as wages, reduces the total cost burden attributed to capital.

When the second stage takes place, the realization of the TFP establishes the kind of shock. If expectations don't overlap with the realization of the TFP, then the shock is understood to be an ambiguity shock. Here our model departs from that of Buera and Moll (2015). In this case, the model heavily relies on the rigidities arising from the first-stage decisions. The reductions of wages and threshold due to an expected reduction of the TFP have positive effects on profits which are offset, *ex ante*, by the reduction of the expected production. However, in this case, the overall effect on production is positive as well, since the reduction in expected production is itself neutralized by the wedge between the realized and expected TFP. In other words the effect on profits is composed by the sum of three positive effects. This determines the large, in absolute value, depart from the steady state. Such an increase in profits have strong implications for the assets accumulation, capital increase keeping up the demand for labor and, then, inducing a crowding-out effect through the threshold. The large, unexpected, increase in assets generates small, but long-lasting, positive effects on both production and wages.

As in Ilut and Schneider (2014), in our setting it is not possible to have positive ambiguity shocks, that is to believe a higher TFP than the realized one. This embeds the possibility to hedge against the worst-case scenario, since it is not allowed to overestimate the future TFP.

To conclude this section, we turn our attention to the business cycles implied by the sequence of ambiguity shocks in our simulated dynamics. We use the band-pass filter in Baxter and King (1999) (with frequency bands 1.5 periods to 8 periods per cycle) to extrapolate the cycles of the sequences of ambiguity and aggregate production. We find that, as in the impulse response at the steady state, there is a match between ambiguity shocks and its effects on production. Although both sequences follow the same path, the variance of ambiguity is roughly three times the variance of aggregate production. This effect reflects the fact that a reduction in ambiguity is followed by a smaller reduction in production. We do the same exercise taking into account wages. In this second case we find a very strong correlation between the series, although now wages cycles oscillate much more than ambiguity.

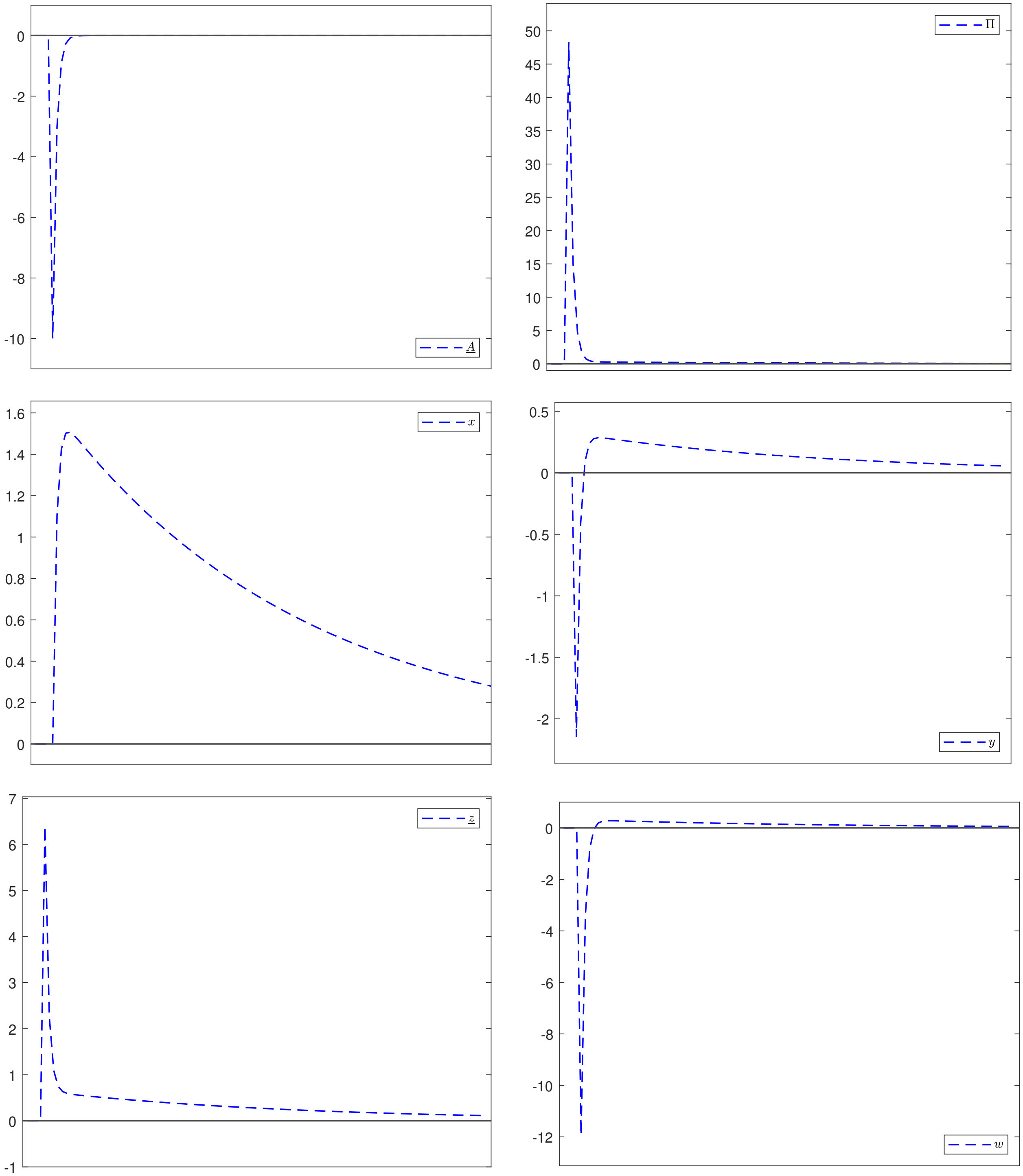

Figure 4: Responses to a 10% negative shock in ambiguity.

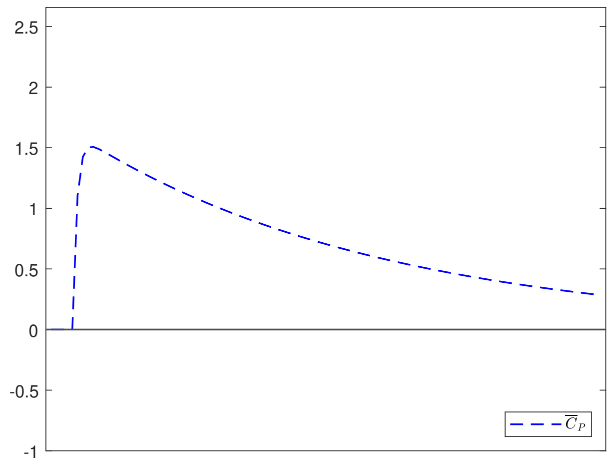

Figure 5: Average consumption responses to a 10% negative shock in ambiguity.

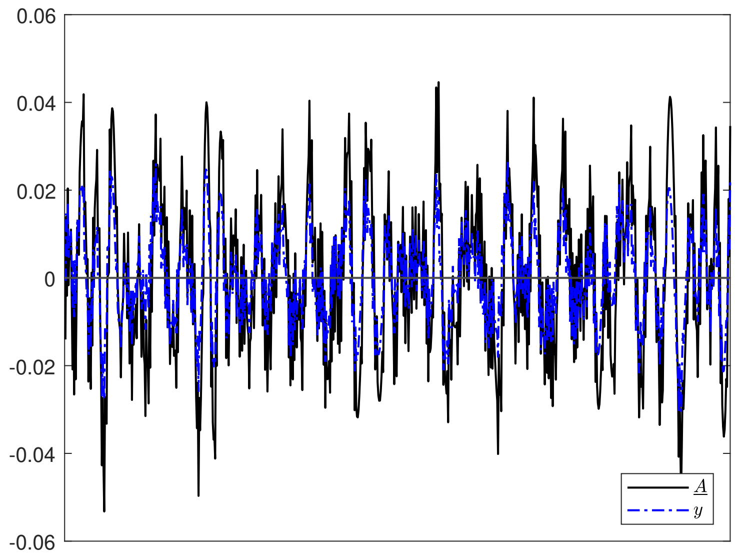

Figure 6: Business cycles, ambiguity and production, simulated data. All series are BK-filtered.

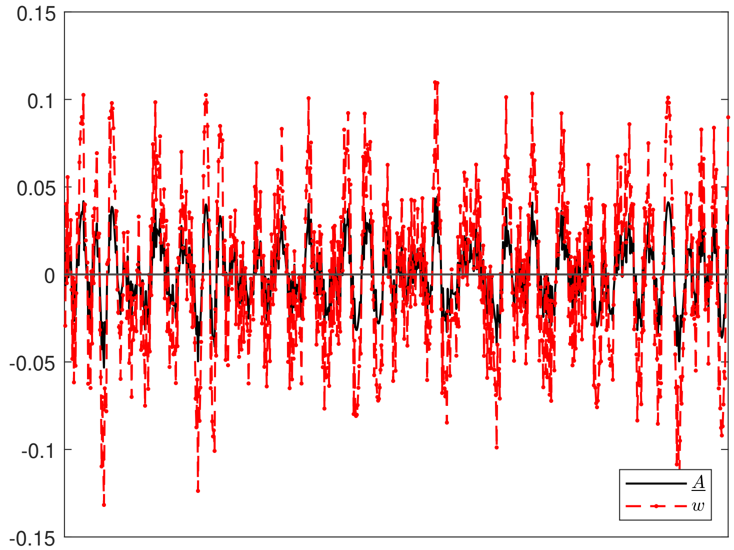

Figure 7: Business cycles, ambiguity and wages, simulated data. All series are BK-filtered.

5 Conclusion

Informational and financial frictions play a crucial role in determining the path of an economy. In this paper we have studied how they contribute to determine business cycles and consumption inequality. We are able to attribute cycles in aggregate production to cycles in ambiguity. Ambiguity proves to have implications both in the short- and in the long-run with a slow return to the steady state in the case it disappears. The mechanism leading to the creation of business cycles involves adjustments as a consequence of the hedging strategy all entrepreneurs put into practice. The hedging strategy proves to be a useful tool for the economy to reduce the effects of ambiguity. In fact, according to our simulation, we find that the cycles in ambiguity oscillate more production cycles. Concerning consumption inequality, ambiguity improves entrepreneurs' condition by allowing over-accumulation of capital. Indeed, our second result shows that the consumption gap widens in favour of entrepreneurs, both in absolute and relative terms, while workers consume less than in an economy without ambiguity.

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Appendices

A Proofs

A.1 Proof of Lemma 1

For any $P \in \mathcal{P}_t$, the maximization problem of an entrepreneur (10) allows to maximize out the choice of labor

$$\frac{\partial \pi^P}{\partial n} = (1-\alpha) A^P \left[\frac{zk}{n}\right]^{\alpha} - w = 0. \tag{39}$$

Substituting back into the profit function

$$\pi^P(z) = A^P \left[\frac{1-\alpha}{w} A^P\right]^{\frac{1-\alpha}{\alpha}} zk - w \left[\frac{1-\alpha}{w} A^P\right]^{\frac{1}{\alpha}} zk - r^* k. \tag{40}$$

$\underline{z}^P$ in (14) can be obtained as the solution for the zero-profit condition,

$$\pi^P\left(\underline{z}^P\right) = A^P \left[\frac{1-\alpha}{w} A^P\right]^{\frac{1-\alpha}{\alpha}} \underline{z}^P k - w \left[\frac{1-\alpha}{w} A^P\right]^{\frac{1}{\alpha}} \underline{z}^P k - r^* k = 0. \tag{41}$$

Conditions (11)-(13) follow automatically, given the implied indicator function. Notice that the choices are at the corner, since the profit function is linear in k.
Since these choices are made *ex ante*, it also urges to assume that contracts cannot be dropped at stage 2, in order to avoid the possibility of contingent contracts and simplify the model. This recovers Lemma 1.

A.2 Proof of Lemma 2

Notice that the leftmost part of (40) is the individual expected production under any $P \in \mathcal{P}_t$,

$$A^P \left[\frac{1-\alpha}{w} A^P\right]^{\frac{1-\alpha}{\alpha}} zk. \tag{42}$$

To find the hedging distribution, it is important to write the utility maximization problem for each entrepreneur, translating the maximization over consumption into one over assets holdings. That is, with the help of (7), (4) can be rewritten as

$$\begin{aligned} U(\mathbf{c}) & \min_{P \in \mathcal{P}_t} \mathbb{E}_P \left[u\left((1+r^*)a_0 + \pi_0^P - a_1\right)\right] \\ & + \min_{P \in \mathcal{P}_t} \mathbb{E}_P \left[\beta\, u\left((1+r^*)a_1^P + \pi_1^P - a_2^P\right)\right] \\ & + \min_{P \in \mathcal{P}_t} \mathbb{E}_P \left[\beta^2\, u\left((1+r^*)a_2^P + \pi_2^P - a_3^P\right)\right] + \dots \end{aligned} \tag{43}$$

The problem collapses to determine $\left\{\min_{P\in\mathcal{P}_t}\pi^P\right\}_{t\in\mathbb{N}}$. Recall that $\mathcal{D}(\underline{A}_t,1)\in\mathcal{P}_t$. Notice that for any $P\in\mathcal{P}_t$, it must be that $A^P\geq\underline{A}_t$ for convexity, thus $\pi^P\geq\pi^{\mathcal{D}}$, and the only case where it holds with strict inequality is when $P\neq\mathcal{D}(\underline{A},1)$. Then, as claimed in Lemma 2, the minimizer is $\mathcal{D}(\underline{A}_t,1)$.

A.3 Proof of Lemma 3

To determine the stream of expected profits it suffices to consider two cases: the present and the future.
At $t=0$, each individual knows their own z_i, still A_0 is ambiguous. Depending on z one either expects $\pi_0^{\mathcal{D}}$ or nothing. From the present point of view the future z is unknown, according to $\mathcal{D}(\underline{A}_t,1)$ the expectation is that to be 1 at every period: if $\{z_{t+q+1}=1\}_{q\in\mathbb{N}}$, the market structure would feature perfect competition, then $\left\{\pi_{t+q+1}^{\mathcal{D}}=0\right\}_{q\in\mathbb{N}}$. More formally, at $t=0$, there is no ambiguity about one's own z, then

$$\min_{P\in\mathcal{P}_t}\pi_0^P=\min_{\mathcal{A}_0}\pi_0^{\mathcal{A}}=\pi_0^P\left(A^{\mathcal{D}}=\underline{A}\right)=\pi_0^{\mathcal{D}}, \tag{44}$$

while at $q\in\mathbb{N}_+$ ambiguity is affecting both A_{t+q} and z_{t+q}, however only the latter matters. In fact, since the two sets of distributions are independent we get

$$\min_{P\in\mathcal{P}_t}\pi_{t+q}^P=\min_{\mathcal{A}_{t+q}}\min_{\mathcal{Z}_{t+q}}\pi_{t+q}^P=\min_{\mathcal{A}_{t+q}}\pi_{t+q}^{\mathcal{A}}(z^{\mathcal{D}}=1)=0. \tag{45}$$

Notice that the problem can be written in terms of $\left\{\pi_{t+q}^{\mathcal{D}}\right\}_{q\in\mathbb{N}}$ only since $\frac{\partial a_{t+q+1}}{\partial\pi_{t+q}^P}>0$, then automatically it holds that $\min_{P\in\mathcal{P}_t}a_{t+q+1}^P=a_{t+q+1}^{\mathcal{D}}$.
The problem has been greatly simplified and it is possible to rewrite (43) as

$$\begin{aligned}U(\mathbf{c})=u&\left((1+r^*)a_0+\pi_0^{\mathcal{D}}-a_1^{\mathcal{D}}\right)\\&+\beta\,u\left((1+r^*)a_1^{\mathcal{D}}-a_2^{\mathcal{D}}\right)\\&+\beta^2\,u\left((1+r^*)a_2^{\mathcal{D}}-a_3^{\mathcal{D}}\right)+\ldots\end{aligned} \tag{46}$$

Notice that we were able to turn a stochastic maximization problem, with respect to $P\in\mathcal{P}_t$, into a deterministic worst-case maximization problem, with $\mathcal{D}(\underline{A}_t,1)$. So far, the problem only depends on whether $\pi_0^{\mathcal{D}}$ is null or not. The recursive structure of the maximization allows to find (17) and (18) as the consumption rules.
The optimal choices must belong to the set by Lemma 1, under the case of $\mathcal{D}(\underline{A}_t,1)$. Then the aggregate labor demand, (21), comes from the aggregation of (12)

$$\int\int_{\underline{z}^P}\left[\frac{(1-\alpha)A^P}{w}\right]^{\frac{1}{\alpha}}z\lambda a\,\mathrm{d}G=\left[\frac{(1-\alpha)\underline{A}}{w}\right]^{\frac{1}{\alpha}}x\int_{\underline{z}}z\lambda\,\mathrm{d}F. \tag{47}$$

Under the measure $\mathcal{D}(\underline{A}_t, 1)$, individual production, (42), becomes

$$\underline{A}\left[\frac{1-\alpha}{w}\underline{A}\right]^{\frac{1-\alpha}{\alpha}} zk. \tag{48}$$

Aggregating, it is possible to write expected aggregate production as a linear function of aggregate assets

$$y^{\mathcal{D}} = \underline{A}\left[\frac{1-\alpha}{w}\underline{A}\right]^{\frac{1-\alpha}{\alpha}} \lambda \underline{z}^{-\eta} x. \tag{49}$$

Combining with (47), one gets (26).

Solve for w in (26) and (16) to obtain (28), the aggregate representation of $\underline{z}$. Substitute (28) back into (26) to have (27). In order to get the share of cost of labor in terms of expected production, it suffices to write expected production in terms of labor, combining (47) and (49), to have (29).

To characterize the expected law of motion of aggregate assets we must also find the aggregate consumption. Aggregate (17) and (18) to get

$$\begin{aligned} C_P &= \int\int_{z\geq\underline{z}} (1-\beta)\left[(1+r^*)a + \pi^{\mathcal{D}}\right] \mathrm{d}G, \\ &= (1-\beta)\left(1 + r^* + \frac{r^*\lambda}{\eta-1}\right)\underline{z}^{-\eta} x, \end{aligned} \tag{50}$$

and

$$\begin{aligned} C_U &= \int\int_{\underline{z}\geq z\geq 1} (1-\beta)(1+r^*)a \,\mathrm{d}G, \\ &= (1-\beta)(1+r^*)\left(1-\underline{z}^{-\eta}\right) x. \end{aligned} \tag{51}$$

Now,

$$\int\int_{z\geq\underline{z}} (1-\beta)\left[(1+r^*)a + \pi^{\mathcal{D}}\right] \mathrm{d}G + \int\int_{\underline{z}\geq z\geq 1} (1-\beta)(1+r^*)a \,\mathrm{d}G, \tag{52}$$

or

$$C_P + C_U = (1-\beta)(1+r^*)x + (1-\beta)\int\int_{z\geq\underline{z}} \pi^{\mathcal{D}} \,\mathrm{d}G, \tag{53}$$

where the rightmost part is

$$\begin{aligned} \int\int_{z\geq\underline{z}} \pi^{\mathcal{D}} \,\mathrm{d}G &= y^{\mathcal{D}} - w\ell - r^{\lambda}\underline{z}^{-\eta} x, \\ &= \alpha y^{\mathcal{D}} - r^*\lambda\underline{z}^{-\eta} x. \end{aligned} \tag{54}$$

That is aggregate consumption is

$$C = (1-\beta)(1+r^*)x + (1-\beta)\left[\alpha y^{\mathcal{D}} - r^*\lambda\underline{z}^{-\eta} x\right]. \tag{55}$$

Finally, the expected law of motion is

$$x_{+1}^{\mathcal{D}} = \beta \int\int_{z \geq \underline{z}} \pi^{\mathcal{D}} \, \mathrm{d}G + \beta(1+r^*)x, \tag{56}$$

that is (30), once (54) is plugged in. This completes Lemma 3.

A.4 Proof of Proposition 1

When stage 2 takes place the real value of A_t is revealed. Since agreements were taken according to the expectation $\underline{A}_t$ and choices cannot be recalibrated, the sunk costs that each entrepreneur will face once the realization happens are low. For instance, with a higher *ex ante* expectation of A_t, wages would have been higher too. The *ex ante* allocation of resources makes realized profits only dependent on the exogenous realization of A_t. Then it is clear that, for a and n fixed and $z \geq \underline{z}$

$$\pi(z) - \pi^{\mathcal{D}}(z) = [A - \underline{A}]\,(zk)^{\alpha} n^{1-\alpha} \propto A - \underline{A}. \tag{57}$$

Negative *ex ante* ambiguity shocks increase realized profits, since this holds for any entrepreneur who committed to produce, this implies that all agreements are respected. Because of higher profits, also the dynamics of wealth accumulation is faster. Then, aggregate realized profits are

$$\begin{aligned}
\int\int_{z \geq \underline{z}} \pi \, \mathrm{d}G &= y - (1-\alpha)y^{\mathcal{D}} - r^* \lambda \underline{z}^{-\eta} x, \\
&= \frac{A_t}{\underline{A}_t} y^{\mathcal{D}} - (1-\alpha)y^{\mathcal{D}} - r^* \lambda \underline{z}^{-\eta} x, \\
&= \left[\frac{A_t}{\underline{A}_t} - 1\right] y^{\mathcal{D}} + \int\int_{z \geq \underline{z}} \pi^{\mathcal{D}} \, \mathrm{d}G.
\end{aligned} \tag{58}$$

Now, the interaction between *ex ante* choices and *ex post* realization hits the dynamics.

$$\begin{aligned}
x_{+1} &= \int\int_{z \geq \underline{z}} \pi \, \mathrm{d}G + \beta(1+r^*)x - (1-\beta) \int\int_{z \geq \underline{z}} \pi^{\mathcal{D}} \, \mathrm{d}G, \\
&= \left[\frac{A_t}{\underline{A}_t} - 1\right] y^{\mathcal{D}} + \int\int_{z \geq \underline{z}} \pi^{\mathcal{D}} \, \mathrm{d}G + \beta(1+r^*)x - (1-\beta) \int\int_{z \geq \underline{z}} \pi^{\mathcal{D}} \, \mathrm{d}G, \\
&= \left[\frac{A_t}{\underline{A}_t} - 1\right] y^{\mathcal{D}} + \beta \int\int_{z \geq \underline{z}} \pi^{\mathcal{D}} \, \mathrm{d}G + \beta(1+r^*)x, \\
&= \left[\frac{A_t}{\underline{A}_t} - 1\right] y^{\mathcal{D}} + x_{+1}^{\mathcal{D}}.
\end{aligned} \tag{59}$$

Where the last line is exactly (34). To recover Proposition 1 it only needs to expand $x_{+1}^{\mathcal{D}}$, using (30), and get (33).

B What if $\underline{z} = 1$

As we have seen in (14) the economy has two states of convergence, depending on its parameter space. To have full market entrance it suffices to set a low interest rate.
In terms of levels, we see that in this case production is higher with a lower capital accumulation and higher wages. This result implies that full participation triggers labor demand reducing capital accumulation and profits, however full participation is enough to compensate for this effects in terms of production.
We model ambiguity shocks as in Section 4 and we find that most of our previous findings hold. By definition, the threshold and the consumption level of non-producing entrepreneurs do not change. However, the consumption level of producing entrepreneurs increases, this is due to the fact that there is no reduction in production since no producers exit the market. Rather, with lower wages in the first period, the over-accumulation effect is triggered and a higher production, with higher consumption, happens in subsequent periods.
Notice that, with full market participation, the long tail of capital over-accumulation is still present, however it is much shorter than with partial market entrance. This effect is determined only partially by higher wages, while it rather comes from the higher level of consumption of producing entrepreneurs, having a long tail itself.
Interestingly, in our simulated analysis, we see that in this case the correlation between cycles is negative, but tends to zero. This reflects the difference in the impulse-response functions we see comparing the two cases. If $\underline{z} = 1$ there is no negative effect in production when the shock takes place. Rather, the economy features a boost in production in subsequent periods. Notice that the business cycles analysis for wages remains unchanged.
To conclude, note by combining these results we see that the positive effects on capital and production are in favour of the former. Combining this fact with (28) this implies that an ambiguity shock, if strong enough, could shift the state from one with $\underline{z} = 1$ to one with $\underline{z} \geq 1$ of the economy.

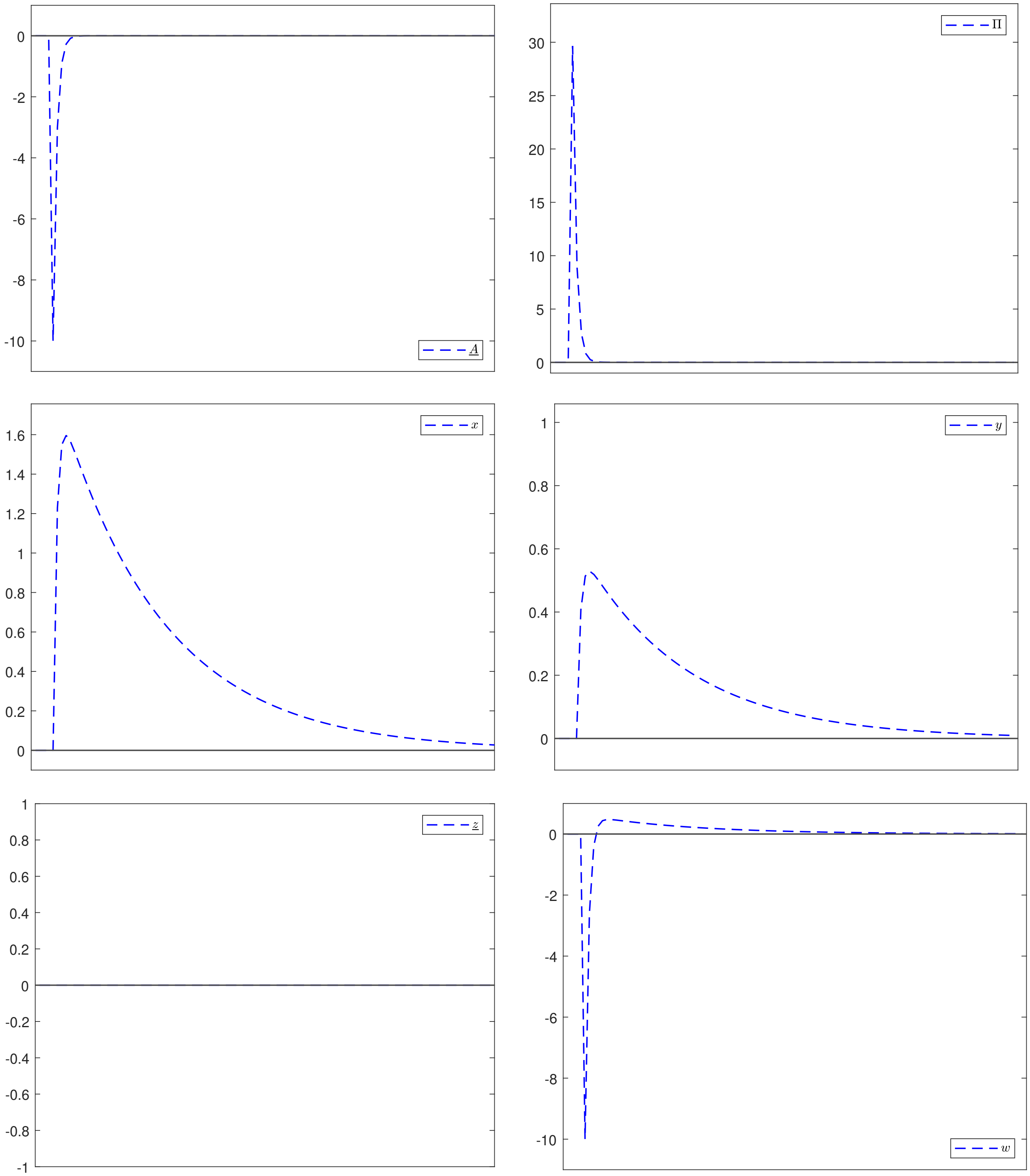

Figure 8: Responses to a 10% negative shock in ambiguity with $\underline{z} = 1$.

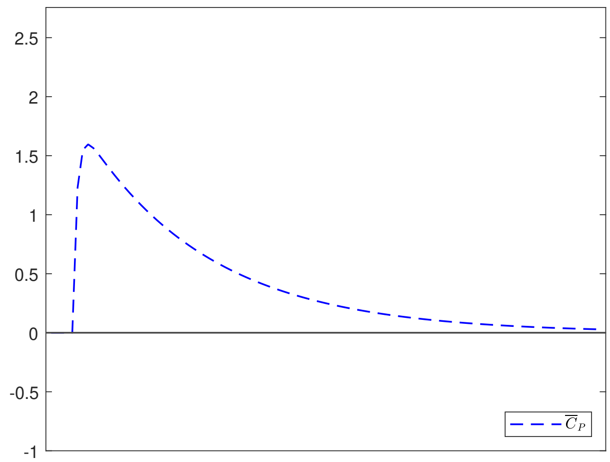

Figure 9: Average consumption responses to a 10% negative shock in ambiguity with $\underline{z} = 1$.

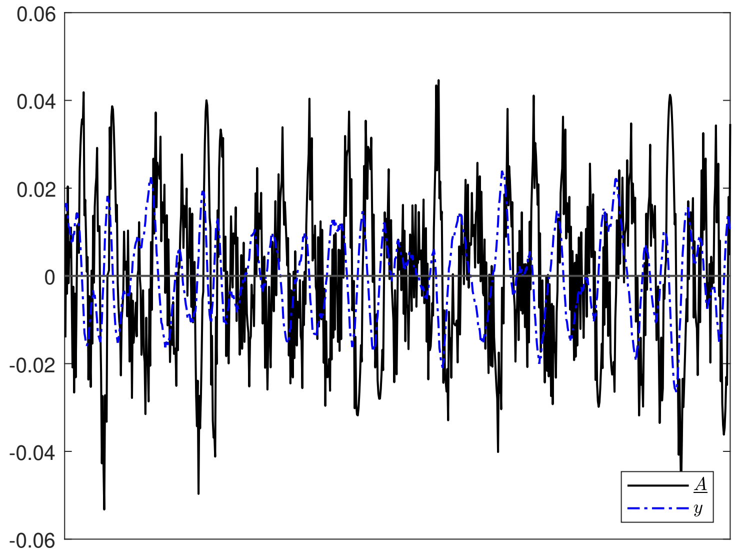

Figure 10: Business cycles, ambiguity and production, simulated data with $\underline{z} = 1$. All series are BK-filtered.

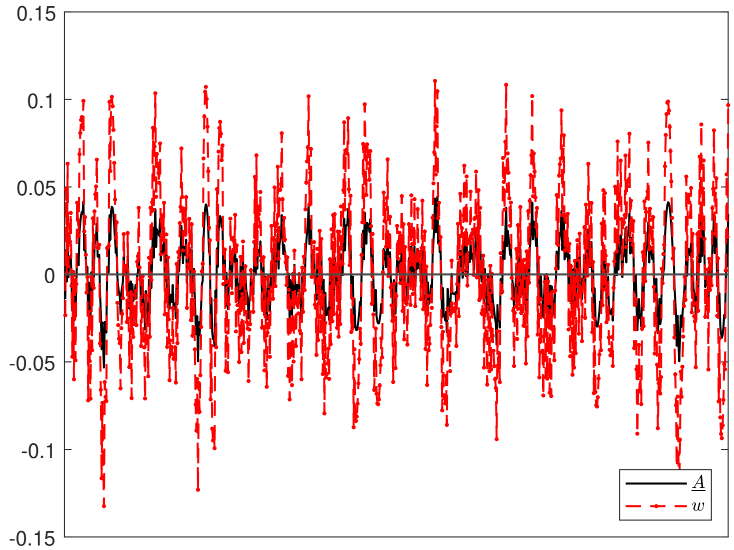

Figure 11: Business cycles, ambiguity and wages, simulated data with $\underline{z} = 1$. All series are BK-filtered.

C Worker side

Following Itskhoki and Moll (2019), assume that workers have a utility of the form

$$U_w = \ln c_{w,t} - \psi \frac{\ell^{1+\varphi}}{1+\varphi}, \tag{60}$$

where φ is the inverse of the Frisch elasticity. The optimal conditions are

$$\frac{\partial U_w}{\partial c_{w,t}} = \mu, \tag{61}$$

$$\frac{\partial U_w}{\partial \ell} = -\mu w. \tag{62}$$

The optimal condition condition (62) translates into the labor supply

$$w = \psi c_{w,t} \ell^{\varphi}, \tag{63}$$

from which, in equilibrium

$$\psi c_{w,t} \ell^{\varphi+1} = (1-\alpha) y^{\mathcal{D}}, \tag{64}$$

expanding $y^{\mathcal{D}}$, workers' consumption is

$$c_{w,t} = \frac{1-\alpha}{\psi} \Theta(\underline{A}) x^{\gamma} \ell^{1-\gamma} \ell^{-\varphi-1}. \tag{65}$$

Combining with the workers' budget constraint, then

$$\begin{aligned} 0 &= c_{w,t} - w\ell, \\ &= \frac{1-\alpha}{\psi}\Theta(\underline{A})x^{\gamma}\ell^{1-\gamma}\ell^{-\varphi-1} - (1-\alpha)\Theta(\underline{A})x^{\gamma}\ell^{1-\gamma}. \end{aligned} \tag{66}$$

That gives $\ell = \left[\frac{1}{\psi}\right]^{\frac{1}{1+\varphi}}$.