Entrepreneurship, growth and productivity with bubbles

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January 17, 2022

Abstract

Entrepreneurship, growth and total factor productivity are larger when asset prices are high and decline during financial crises. We explain these facts using a growth model with bubbles in which individuals have heterogeneous wages and returns on productive investment. Heterogeneity separates individuals between savers and entrepreneurs. Savers buy financial assets, which are deposits or a financial bubble. Entrepreneurs incur in a start-up cost and borrow to invest in productive capital. The bubble provides liquidities to credit-constrained entrepreneurs. These liquidities increase investment and entrepreneurship. Finally, the bubble may increase productivity when the return on investment is correlated with wages.

JEL classification: E22; E44; G12.

Keywords: Bubble, entrepreneurship, growth, productivity.

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1 Introduction

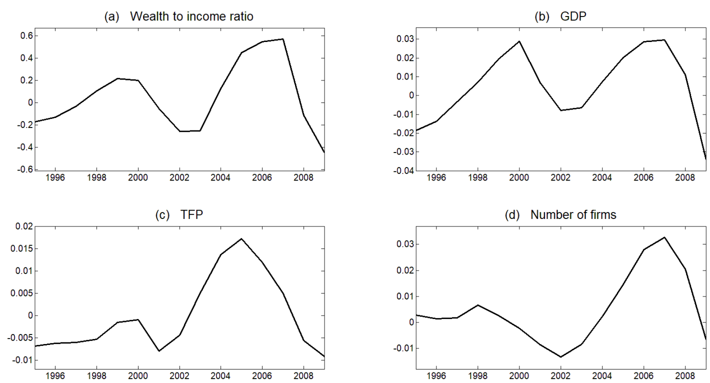
After the Great Recession, there has been renewed interest in analyzing the growth effects of financial bubbles. In this paper, we contribute to this literature and we study the effect of financial bubbles on entrepreneurship, growth and productivity. Figure 1 shows the time series of these variables for the US economy in the period 1995-2009. The different panels show the business cycle component of wealth to income ratio, logarithm of gross domestic product (GDP), total factor productivity (TFP) and logarithm of the number of firms. In panel (a), we observe two periods in which wealth relative to income increases substantially and after declines sharply. As explained by Martin and Ventura (2012), these large fluctuations in wealth are driven by asset price fluctuations and no fundamental seems to explain the fluctuations of asset prices during these periods. As a result, these two periods have been considered as examples of financial bubbles. The rest of panels in Figure 1 show that these large fluctuations in wealth are closely related to fluctuations in GDP, in TFP and in the number of firms. We observe that these three variables are large when wealth is large and they decrease when wealth declines. In fact, the correlations between the business cycle component of the wealth to income ratio and that of GDP, TFP and the number of firms are, respectively, 0.91, 0.69 and 0.76.[1] Clearly, correlations are positive, very large and significative.

The literature provides adding empirical supports to these findings using both aggregate and firm-level data for different countries. Using aggregate data, the literature distinguishes two facts. First, Campbell (1999), among many others, show that asset price volatility is highly procyclical. We follow the literature on financial bubbles and interpret asset price growth as the result of a bubble and the reduction of asset prices as the result of a bubble burst. According to this interpretation, growth is larger when there is a bubble. Caballero *et al.* (2006) and Martin and Ventura (2012) provide convincing evidence on this relationship. They identify periods during which, according to most of the literature, there is a bubble and show that growth

[1] The value of the correlations is similar if we compute them using the growth rate of the variables instead of the business cycle component. We also obtain similar findings when we consider the period 1978-2018, for which data is available.

is large in these periods and it declines when the bubble bursts. Second, Basu and Fernald (2001), Basu et al. (2006) and Field (2010) show that total factor productivity (TFP) is procyclical.[2] Since bubbles are also procyclical, this second evidence suggests that TFP is large when there is a bubble and declines when it bursts. This is consistent with findings obtained by Meza and Quintin (2005), Pratap and Urrutia (2012), Queralto (2011) and Tang (2017), who find that TFP fell during East Asian, Mexican and Argentine financial crises in the 1990s and in Spain during the Great Recession.

Figure 1. US time series



Note. The figure plots the cyclical component of the wealth to income ratio, the logarithm of GDP, TFP and the logarithm of the number of firms obtained using the Hodrick-Prescott filter with a smoothing parameter of 100. We use annual data for the US in the period 1995-2009. Wealth to income ratio is households and nonprofit organizations net worth as a percentage of disposable personal income and is obtained from Federal Reserve Bank of St. Louis. GDP is real GDP at chained PPPs and TFP is at constant national prices, both obtained from the Penn World Table 10.0. The number of firms is obtained from the Business Dynamics Statistics from the US Census Bureau.

The literature also distinguishes two facts using firm-level data. First, Koellin-ger and Thurik (2012) show that entrepreneurship is procyclical.[3] Since bubbles are also procyclical, this finding suggests that the number of entrepreneurs increases during a

[2]Fernald and Wang (2016) confirm that TFP is procyclical, although they also show that after mid-1980s the TFP became less procyclical.

[3]Koellinger and Thurik (2012) show a positive correlation between entrepreneurship and deviations of GDP from trend for a cross-country panel of 22 OECD countries for the period 1972 to 2007. Bilbiie et al. (2012), Campbell (1998), and Clementi et al. (2016) show that, in the US, firms entry is procyclical, while exit is countercyclical.

bubble and declines when it bursts. This has been confirmed by Klapper and Love (2011) and Tian (2018), who show that the number of entrepreneurs falls during the Great Recession. Second, Bartelsman and Doms (2000), Lee and Mukoyama (2015) and Tian (2018) show that new firms are smaller and less productive than incumbents and that these differences are more pronounced in booms than in recessions. Therefore, periods with financial bubbles correspond to periods in which the number of firms increases, but these new firms are less productive and smaller in size.

We show that these four facts can be explained in an overlapping generations (OLG) model with the following characteristics. First, we assume that individuals live for three periods and can invest in productive capital only in the second period of life, whereas part of the labor income is obtained in the first period. In the first period, individuals save or borrow through two different financial assets: a deposit (or a credit) and a financial bubble, which takes the form of a purely speculative asset. In the second period, individuals face a borrowing constraint that limits investment. Farhi and Tirole (2012) show that financial bubbles can be sustained in the equilibrium of a model with these characteristics and these bubbles, by increasing the savings devoted to the demand of financial assets, provide the liquidities needed to invest when individuals are credit constrained. This is the liquidity effect that increases investment and growth.[4] Second, we extend the model by assuming a continuous distribution of abilities in the population, as in Kunieda and Shibata (2016). These abilities are an individual-specific productivity shock that determines the return of productive investment. Those individuals with a large return become entrepreneurs that use the financial assets to borrow and invest in productive capital, whereas the rest are savers who accumulate financial assets. Third, we assume that abilities

[4] The literature distinguishes between two growth-enhancing roles of the bubbles. One is the liquidity role of the bubble: agents hold at the beginning of the period the bubble and sell it to increase their productive investment (Kocherlakota, 2009; Farhi and Tirole, 2012; Martin and Ventura, 2012; Hirano and Yanagawa, 2017; Miao and Wang, 2018).The other one is the collateral role of the bubble: agents buy the bubble to increase their possibilities to borrow and use these loans to invest in capital (Kocherlakota, 2009; Martin and Ventura, 2016; Miao and Wang, 2018). In Clain-Chamosset-Yvrad et al. (2020) we show that in the absence of uncertainty both roles are identical. Accordingly, in this paper we do not distinguish them and we simply refer to this growth enhancing effect of the bubble as the liquidity effect.

also determine the productivity of workers. As a result, wages are heterogeneous. Finally, we assume that individuals incur a start-up cost to be entrepreneurs. The introduction of this cost and wage heterogeneity are the novelties of this model. We show that they introduce new economic mechanisms that are crucial to explain the aforementioned facts.

The equilibrium of this model can converge to two different steady states: a bubbly steady state in which financial assets are deposits and the speculative asset, and a bubbleless steady state in which the only financial assets are deposits. We show that the return of financial assets is larger in the equilibrium with bubbles, which is a consequence of the larger demand of financial assets in this equilibrium. We also show that if a bubbly steady state exists, then the return of financial assets equals the growth rate in this steady state and it is lower in the bubbleless steady state. These results coincide with those obtained by Tirole (1985) and Grossman and Yanagawa (1993) in models in which individuals that belong to the same generation are identical.

We compare the two steady states to study the effect of the bubble on the number of entrepreneurs, on growth and on productivity. We first show that two opposite mechanisms determine the effect of the bubble on the composition of the population between entrepreneurs and savers. On the one hand, in a bubbly steady state the return of financial assets is larger, which implies that more individuals choose to be savers, as in Kunieda and Shibata (2016). On the other hand, the liquidities provided by the bubble make adult individuals wealthier, which facilitates that more individuals can afford the start-up cost. Therefore, the number of entrepreneurs is larger in the bubbly steady state when this cost mechanism dominates. This mechanism, that can be interpreted as the extensive margin of the liquidity effect, explains that we observe a larger number of entrepreneurs during bubbly periods. Moreover, we show that these new entrepreneurs are less productive and invest less than existing entrepreneurs, which is also in line with firm level evidence.[5]

We also show that the bubble affects growth through three distinct effects: the liquidity, leverage and composition effects of the bubble. First, as in Farhi and Ti-

[5] This result is related to Aghion et al. (2019) that shows that better credit access allows less efficient firms to remain longer in the market.

role (2012), the liquidity effect promotes growth, because the bubble provides the liquidities that credit constrained entrepreneurs need to invest. Second, the larger return of financial assets in the bubbly steady state reduces the amount of credit that can be obtained using productive investment as collateral. As a result, productive investment decreases. This is the leverage effect that reduces growth. Finally, the composition effect or the extensive margin of the bubble is a contribution of this paper. We show that this effect promotes growth when the bubble increases the number of entrepreneurs.

We finally show that the previous three effects also modify TFP, which is equal to the average return of productive investment. The composition effect decreases TFP when the number of entrepreneurs increases, since new entrepreneurs have lower productivities. The leverage effect reduces TFP, since the increase in the return of financial assets causes a larger decline of the investment of more productive entrepreneurs and, as a result, the average return of productive investment decreases. Finally, the liquidity effect of the bubble increases TFP when the productivity of investment is more correlated with the wages in the first period of life than with the wages in the second period. When this happens, the bubble provides more liquidities to more productive entrepreneurs. As a consequence, it further increases the investment of highly productive entrepreneurs, which explains that the liquidity effect increases TFP.

We conclude that the bubble can increase the number of entrepreneurs, growth and TFP, even though new entrepreneurs are less productive. Therefore, by adding the composition effect or extensive margin of financial bubbles, this model explains the facts obtained using both aggregate and firm level data.

This paper is related to two strands of the literature. First, it is related to findings in the literature that studies the effect of financial development on growth and TFP. This literature interprets financial development as access to external financing that, together with self-financing, is used to invest (see Cooley and Quadrini, 2001; Midrigan and Xu, 2014). In this literature, self-financing substitutes external financing. As a result, Moll (2014) shows that external financing cause a smaller increase in TFP when idiosyncratic shocks are persistent, since more productive entrepreneurs have access to larger self-financing. We also analyze how the interaction between the persistence of idiosyncratic shocks and external financing affects TFP. In our model, the persistence of shocks is measured by the correlation between wages of the young and investment productivity of the adult, self-financing corresponds to the savings of the young individuals who will be entrepreneurs in the following period and we consider two sources of external financing: credit and the bubble. The bubble introduces significant differences. Unlike Moll (2014), we show that when this correlation is large, the external financing, introduced in our framework by the bubble, positively affects TFP. The reason for this different finding is that the bubble increases the returns of the savings of the young individuals and, hence, it enlarges the effect of self-financing on investment. Thus, self-financing complements the bubble, whereas it substitutes credit.

Second, it is related to the literature on financial bubbles. Very few papers in this literature consider the effect of financial bubbles on the number of entrepreneurs. One exception is Kunieda and Shibata (2016), who study an economy in which individuals decide between being savers or entrepreneurs.[6] However, they do not introduce the start-up cost and, hence, the bubble reduces the number of entrepreneurs. In any case, the literature on financial bubbles has mainly studied the effect of bubbles on growth and productivity when the number of entrepreneurs is constant. This literature has studied the growth effects of bubbles since the seminal papers by Tirole (1985) and Grossman and Yanagawa (1993). In these papers, the introduction of a speculative asset without fundamental value, a financial bubble, reduces productive investment and growth. More recent literature has shown that if individuals are heterogeneous and face credit constraints, then bubbles can promote growth, which is more in line with evidence. For instance, Farhi and Tirole (2012) and Martin and Ventura (2012) show that financial bubbles may promote growth when they provide to credit constrained entrepreneurs the liquidities needed to invest.[7] Some papers

[6]Kunieda (2008) considers the same mechanism in the context of an OLG model and Kunieda (2014) also introduces this mechanism to study the effect of bubbles on growth in a model of perpetual youth.

[7]There are many other examples of models with bubbles and heterogeneous individuals. For instance, Bengui and Phan (2018) and Graczyk and Phan (2021) consider that individuals have different endowments, which separates individuals between borrowers and lenders. This distinction

have also studied the effect of bubbles on TFP. In particular, Miao and Wang (2012) show that if bubbles increase investment of more productive entrepreneurs relative to less productive ones then TFP is larger with bubbles, which is in line with evidence. Hirano and Yanagawa (2017) obtain a similar conclusion in an endogenous growth model.

The liquidity and leverage effects in our paper, that correspond to the intensive margin of the bubble, are based on the same mechanisms that the literature on financial bubbles has introduced to explain the effects of bubbles on growth and on productivity. Therefore, we contribute to this literature by adding the extensive margin or composition effect of the bubble. Using numerical examples, we show that this margin has a sizeable effect on both growth and productivity. This result is consistent with findings in the literature on firm dynamics that has shown that the extensive margin is an important channel through which financial development affects productivity (see Midrigan and Xu, 2014; Buera, et al., 2011; Jeong and Townsend, 2007).

This paper is organized as follows. Section 2 presents the model. Section 3 characterizes the equilibrium. Section 4 analyzes the effect of bubbles on entrepreneurship, growth and productivity, including a numerical illustration and a comparison with the literature. Concluding remarks are provided in Section 5, while some technical details are relegated to an online appendix.

2 Model

We consider a discrete time overlapping generations model ($t = 1, 2, ...$) populated by individuals that can be entrepreneurs or savers.

2.1 Production

We consider an aggregate production function that relates final output, y_t, with aggregate capital, k_t, and efficiency units of labor, l_t. We assume that workers have

is also in Basco (2016) and in Kocherlakota (2009) in a model of infinitely lived agents. In contrast, Hillebrand (2018) distinguishes between three groups of individuals: savers, entrepreneurs and semi-entrepreneurs.

heterogeneous efficiency units of labor. The variable l_t measures the aggregate efficiency units of labor supplied by all workers. In addition, an externality associated to the average capital to labor ratio, $\overline{a}_t$, increases labor productivity. Therefore, the aggregate production function is

$$y_t = F(k_t, \bar{a}_t l_t).$$

This production function has the usual neoclassical properties, that is, it is a strictly increasing and concave production function satisfying the Inada conditions and is homogeneous of degree one with respect to its two arguments. Profit maximization under perfect competition implies that the wage w_t per efficiency unit and the return of capital q_t are given by

$$w_t = F_2(k_t, \bar{a}_t l_t)\bar{a}_t, \tag{1}$$

$$q_t = F_1(k_t, \bar{a}_t l_t). \tag{2}$$

We will consider only symmetric equilibria for which $\overline{a}_t = a_t$, where $a_t \equiv k_t/l_t$. Using (1) and (2), we deduce that the wage per efficiency unit, the return of capital and aggregate production at an equilibrium satisfy

$$w_t = (1-s)Aa_t, \tag{3}$$

$$q_t = sA, \tag{4}$$

$$y_t = Ak_t, \tag{5}$$

where $s \equiv F_1(1,1)/F(1,1) \in (0,1)$ is the capital income share and $A \equiv F(1,1) > 0$.

2.2 Individuals

The economy is populated by individuals $i \in [0, N]$ that live for three periods: young, adult and old. The mass of individuals in each generation, N, is constant.

Young and adult individuals work and obtain wages $w^i_{y,t}$ and $w^i_{a,t+1}$, respectively. Young individuals consume $c^i_{1,t}$ and save using two different financial assets: a speculative asset, $b^i_{1,t}$, with return $R_{1,t+1}$ and a deposit, $d^i_{1,t}$, with return $R_{d,t+1}$. Adult individuals consume $c^i_{2,t+1}$, may invest κ^i_{t+2} in productive capital and save $b^i_{2,t+1}$ in the speculative asset and $d^i_{2,t+1}$ in the deposit. We denote savers to those individuals who only save through financial assets and entrepreneurs to those other individuals who also invest in productive capital. Entrepreneurs obtain a return of productive investment that is individually specific and equal to q^i_{t+2}.[8] Capital totally depreciates after a period. The return of the speculative asset purchased by adult individuals is $R_{2,t+2}$, which is a priori different from the return of the speculative asset purchased by young individuals. These pure speculative assets are the financial bubbles. Finally, old individuals do not work, obtain the return of the different investments made when adult and consume $c^i_{3,t+2}$. Accordingly, the budget constraints of the young, adult and old individuals are, respectively:

$$c^i_{1,t} + d^i_{1,t} + b^i_{1,t} = w^i_{y,t}, \tag{6}$$

$$c^i_{2,t+1} + \kappa^i_{t+2} + d^i_{2,t+1} + b^i_{2,t+1} = w^i_{a,t+1} + R_{d,t+1} d^i_{1,t} + R_{1,t+1} b^i_{1,t}, \tag{7}$$

$$c^i_{3,t+2} = q^i_{t+2}\kappa^i_{t+2} + R_{d,t+2} d^i_{2,t+1} + R_{2,t+2} b^i_{2,t+1}. \tag{8}$$

Financial assets are used to borrow when they take negative values. A negative value of the speculative asset implies that individuals short sell this asset, whereas a negative deposit is a credit. Therefore, adult individuals that borrow have access to two different sources of external financing: the credit and the bubble. These individuals face the following constraint:[9]

$$-R_{d,t+2} d^i_{2,t+1} \leq R_{2,t+2} b^i_{2,t+1} + \theta q^i_{t+2}\kappa^i_{t+2}. \tag{9}$$

This credit constraint ensures a strictly positive wealth in the last period of life and implies that the cost of the credit, $-R_{d,t+2} d^i_{2,t+1}$, is limited by the value of the bubble in the last period and by the return of productive investment. Therefore, both the bubble and a fraction $\theta \in [0, 1)$ of productive investment are used as collateral. The parameter θ measures the degree of pledgeability of productive investment and, hence, it is a measure of financial development.

The returns of the different financial assets have different interpretations. It is an interest factor for deposits, while it is the growth of the price for the speculative

[8] This individually specific return of investment is consistent with recent evidence that shows that the return of investment increases with the wealth of the investor (see Fagereng, et al., 2020).

[9] A similar credit constraint was introduced by Martin and Ventura (2016), and by Kocherlakota (2009) in the particular case in which $\theta = 0$.

asset. Despite this different interpretation, from the budget and credit constraints we observe that the financial assets are perfect substitutes and, therefore, their returns coincide, i.e. $R_{d,t+1} = R_{1,t+1} = R_{2,t+1}$. We denote by R_{t+1} this common return of financial assets.

Preferences of an individual i born in period t are represented by the following utility function:

$$\alpha \ln(c^i_{1,t}) + \beta \ln(c^i_{2,t+1} - f^i_{t+1}) + \gamma \ln(c^i_{3,t+2}), \tag{10}$$

where α, β and γ are positive preference parameters that satisfy $\alpha + \beta + \gamma = 1$, and f^i_{t+1} is a start-up cost that individuals must pay to be entrepreneurs.

Adult individuals that decide to be entrepreneurs spend time searching for productive investment opportunities, which introduces a start-up cost that takes the form of a time cost. This time cost reduces leisure time and, therefore, causes a utility loss. Since it is a time cost, we assume that it is proportional to the wage of adult individuals, $w^i_{a,t+1}$. More specifically, the start-up cost is $f^i_{t+1} = \xi w^i_{a,t+1} > 0$ if the individual is an entrepreneur, whereas $f^i_{t+1} = 0$ if she is not.[10] The parameter $\xi \in (0, 1)$ is the start-up cost rate that measures the fraction of time spend searching for investment opportunities. The additive form of the utility function used to introduce the start-up cost has the advantage that the cost can also be interpreted as a reduction of the adult individuals' employment or as a cost in terms of consumption goods. In fact, the solution of the individuals problem and the mechanism that determines the number of entrepreneurs are identical under these different interpretations.[11] Finally, it is important to underline that the start-up cost does not depend on the amount of entrepreneurs' investment. As a result, this cost introduces a discontinuity in the

[10] Poschke (2013), among many others, has also assumed that the cost of becoming an entrepreneur is an opportunity cost in terms of forgone wages. In Chatterjee et al. (1993) and many others, this opportunity cost directly causes a utility loss.

[11] We have interpreted the start-up cost as a reduction in the time devoted to leisure. However, we could have also interpreted it as a reduction in the time available to work or as an expenditure. To see that the solution of the consumers' problem is identical under these different interpretations of the cost, it is enough to define consumption when adult as $\widetilde{c}^i_{2,t+1} = c^i_{2,t+1} - f^i_{t+1}$. Rewriting the adults' budget constraint using $\widetilde{c}^i_{2,t+1}$, it is immediate to see that adults' labor income is $(1 - \xi)\, w^i_{a,t+1}$, which is consistent with the cost implying a reduction in the time devoted to work. It is also consistent with an increase in the expenditures of $\xi w^i_{a,t+1}$.

utility function.

We finish the description of the model by introducing heterogeneity. We assume that individuals are heterogeneous in their innate abilities, δ^i. In every generation, δ^i follows a time invariant and continuously differentiable cumulative distribution function $F\left(\delta^i\right)$ with support $\delta^i \in (\delta_{\min}, \delta_{\max})$. These abilities are an individual-specific productivity shock that determines the return of productive investment and wages. On the one hand, an individual i that invests κ^i_{t+2} units when adult obtains $\delta^i \kappa^i_{t+2}$ units of productive capital when old. Therefore, the return of investment is $q^i_{t+2} = q_{t+2}\delta^i$, where $q_{t+2} = sA$ is the constant return of capital. It follows that the return of investment is perfectly correlated with abilities.

On the other hand, wages of the young individuals satisfy: $w^i_{y,t} = \left(\delta^i\right)^{v_1} w_t$, where w_t is the wage per efficiency unit, $\left(\delta^i\right)^{v_1}$ measures the efficiency units of a young individual i and $v_1 \in [0,1]$ measures the correlation between wages of young individuals and abilities. Since abilities are perfectly correlated with the return of productive investment, v_1 also measures the correlation between this return and the wages of young individuals. Adult individuals wages are also an increasing function of abilities satisfying: $w^i_{a,t+1} = \phi\left(\delta^i\right)^{v_2} w_{t+1}$, where $\phi\left(\delta^i\right)^{v_2}$ measures the efficiency units of labor of an adult individual i. The parameter $\phi \geq 1$ measures the common increase of wages in adulthood associated to accumulated skills and $v_2 \geq 0$ measures the correlation between wages of adult individuals and the return of investment. We assume that young individuals' wages are more correlated with innate abilities; i.e. $v_1 \geq v_2$. Assumption A groups all these assumptions.

Assumption A. $v_j \in [0,1]$, $j = 1, 2$, $v_1 \geq v_2$, $\phi \geq 1$ and $\delta_{\min} = 1$.

As follows from Assumption A, we also assume that $\delta_{\min} = 1$, which implies that the return of investment of the individual with the lowest ability equals the return of aggregate capital and this individual is endowed with one efficiency unit of labor when young and ϕ units when adult.

2.3 Individuals' decisions

To characterize individual's decisions on both consumption and investment, we must take into account that the start-up cost introduces a discontinuity in the utility function. Therefore, we solve the individuals' problem by backward induction following a two-step procedure. First, we obtain the individuals' optimal demands of both consumption and assets that maximize (10) subject to the budget constraints (6)-(8), the credit constraint (9) and a non-negativity constraint on investment, $\kappa^i_{t+2} \geq 0$. By solving this maximization problem, we distinguish between two groups of individuals: savers and entrepreneurs. Savers are those individuals that only invest in financial assets, whereas entrepreneurs are those individuals that pay the start-up cost and invest in productive capital. In a second step, we use the individuals' consumption demands to obtain the indirect utility function of both savers and entrepreneurs. We compare these indirect utility functions to determine the amount of entrepreneurs. Details are given in the online appendix.

We consider first the optimal decisions of savers. They are not credit constrained and their consumption decisions are determined by the following first order conditions:

$$c^{i,S}_{2,t+1} = (\beta/\alpha)\, R_{t+1} c^{i,S}_{1,t}, \tag{11}$$

$$c^{i,S}_{3,t+2} = (\gamma/\beta)\, R_{t+2} c^{i,S}_{2,t+1}, \tag{12}$$

where $c^{i,S}_{1,t}$, $c^{i,S}_{2,t+1}$ and $c^{i,S}_{3,t+2}$ denote, respectively, the consumption of young, adult and old savers.[12] Savers only invest in financial assets, the deposit and the bubble. We denote the value of financial assets owned by young and adult savers as $x^{i,S}_{1,t} = b^{i,S}_{1,t} + d^{i,S}_{1,t}$ and $x^{i,S}_{2,t+1} = b^{i,S}_{2,t+1} + d^{i,S}_{2,t+1}$. We obtain, from the budget constraints (6)-(8) and from the first order conditions (11) and (12), that the demands of financial assets satisfy:

$$x^{i,S}_{1,t} = (\beta + \gamma)\, w^i_{y,t} - \alpha \frac{w^i_{a,t+1}}{R_{t+1}}, \tag{13}$$

$$x^{i,S}_{2,t+1} = \gamma (R_{t+1} w^i_{y,t} + w^i_{a,t+1}). \tag{14}$$

We next consider the optimal decisions of entrepreneurs. They are credit constrained and their consumption and investment decisions are determined by the following first order conditions:

$$c^{i,E}_{2,t+1} - f^i_{t+1} = (\beta/\alpha)\, R_{t+1} c^{i,E}_{1,t}, \tag{15}$$

$$c^{i,E}_{3,t+2} = (\gamma/\beta)\,(1-\theta)\,\psi^i_{t+2} q^i_{t+2} (c^{i,E}_{2,t+1} - f^i_{t+1}), \tag{16}$$

[12] The superscript S identifies the optimal decisions of savers, whereas the superscript E identifies the optimal decisions of entrepreneurs.

where $c_{1t}^{i,E}$, $c_{2t+1}^{i,E}$ and $c_{3t+2}^{i,E}$ denote, respectively, the consumption of young, adult and old entrepreneurs and $\psi_{t+2}^{i}=1/\left(1-\theta q_{t+2}^{i}/R_{t+2}\right)$ is the multiplier of productive investment.

We define the value of financial assets owned by young and adult entrepreneurs as $x_{1,t}^{i,E}=b_{1,t}^{i,E}+d_{1,t}^{i,E}$ and $x_{2,t+1}^{i,E}=b_{2,t+1}^{i,E}+d_{2,t+1}^{i,E}$ and we use the binding credit constraint, the budget constraints (6)-(8) and the first order conditions (15) and (16) to obtain

$$x_{1,t}^{i,E}=\left(\beta+\gamma\right)w_{y,t}^{i}-\alpha\frac{w_{a,t+1}^{i}-f_{t+1}^{i}}{R_{t+1}}, \tag{17}$$

$$x_{2,t+1}^{i,E}=-\left(\frac{\theta\gamma q_{t+2}^{i}}{R_{t+2}}\right)\psi_{t+2}^{i}\left(R_{t+1}w_{y,t}^{i}+w_{a,t+1}^{i}-f_{t+1}^{i}\right), \tag{18}$$

and the amount of productive investment

$$\kappa_{t+2}^{i}=\gamma\psi_{t+2}^{i}\left(R_{t+1}w_{y,t}^{i}+w_{a,t+1}^{i}-f_{t+1}^{i}\right). \tag{19}$$

Entrepreneurs are not credit constrained when young and, therefore, they use the financial assets to smooth consumption between their first two periods of life. Instead, they are credit constrained when adult. Therefore, they smooth consumption between the second and third periods of life using productive investment. As follows from (19), productive investment depends on the multiplier of productive investment, ψ_{t+2}^{i}. This multiplier arises because the return of productive investment is used as collateral (see the borrowing constraint (9)). Note that the multiplier must be positive and this requires $q_{t+2}^{i}<R_{t+2}/\theta$ for all individuals. This constraint prevents entrepreneurs from borrowing an infinite amount, which could not happen in equilibrium. In the following section, we show that this constraint introduces an upper bound on the value of abilities.

The investment multiplier of more productive entrepreneurs is larger, since these entrepreneurs obtain a larger return of investment. As a result, more productive entrepreneurs obtain a larger credit and invest a larger amount in productive capital when $\theta>0$. Therefore, the interaction between the credit constraint and the heterogeneous returns of productive investment is one determinant of the different amounts invested by entrepreneurs. The other determinant is labor income differences. As follows from (19), investment in productive capital increases with the adults' wealth, defined as the present value of labor income net of start-up costs. Since entrepreneurs with larger innate abilities benefit from a larger investment multiplier and have larger wages, they invest more.

We next consider the individual decision between being a saver or an entrepreneur. In the online appendix, we show that an individual that is adult in period t obtains a larger utility as entrepreneur when $q_{t+1}^i > R_{t+1}/\omega_t^i$, where

$$\omega_t^i = \left(1 - \frac{f_t^i}{w_{a,t}^i + R_t w_{y,t-1}^i}\right)^{\frac{1}{\gamma}} (1-\theta) + \theta. \tag{20}$$

Since $f_t^i \leq w_{a,t}^i$ and $\theta < 1$, it follows that $\omega_t^i \in (\theta, 1)$ when $f_t^i > 0$. Therefore, an individual becomes an entrepreneur when the return of productive investment is strictly larger than the return of financial assets; that is, $q_{t+1}^i > R_{t+1}$. In contrast, in the absence of a start-up cost, $\omega_t^i = 1$ and, therefore, an individual with $q_{t+1}^i = R_{t+1}$ is indifferent between being an entrepreneur or a saver. We reached the same conclusion if we consider the limiting case in which the degree of pledgeability equals one. It follows that the interaction between the start-up cost and financial frictions explain that $q_{t+1}^i > R_{t+1}$. More precisely, a larger start-up cost or a smaller degree of pledgeability increase the minimum return of productive investment necessary to be an entrepreneur and, hence, reduce the number of entrepreneurs.

Using (3) and (20), we rewrite ω_t^i as the following increasing function of both δ^i and the ratio between the interest factor and the growth factor, $z_t \equiv R_t/g_t$, where $g_t \equiv a_t/a_{t-1}$:

$$\omega_t^i \equiv \omega\left(\delta^i, z_t\right) = \left(1 - \frac{\xi\phi\left(\delta^i\right)^{v_2}}{\phi\left(\delta^i\right)^{v_2} + \left(\delta^i\right)^{v_1} z_t}\right)^{\frac{1}{\gamma}} (1-\theta) + \theta. \tag{21}$$

As shown in (20), the effect of the start-up cost on ω_t^i is determined by the ratio $f_t^i/\left(w_{a,t}^i + R_t w_{y,t-1}^i\right)$. This ratio measures the cost as a fraction of the present value of labor income in the second period of life. Since young individuals obtain wages in the first period and $f_t^i = \xi w_{a,t}^i$, this ratio decreases with the interest factor and increases with the growth rate of wages, which in this model coincides with the growth rate of a_t. As a result, the ratio between the start-up cost and the present value of labor income decreases with the ratio z_t. This explains that ω_t^i increases with z_t. In other words, the ratio z_t determines the effect of first period wages on adult's wealth.

Hence, when z_t increases, adult's wealth increases and the cost as a fraction of this wealth decreases.

We next determine the number of entrepreneurs. To this end, we define by $\overline{\delta}_t$ the ability of the marginal individual that in period t is indifferent between investing in financial assets or in productive capital. This individual satisfies that $q_{t+1}^i = R_{t+1}/\omega_t^i$ and, since $q_{t+1}^i = \delta^i sA$, we obtain that $\overline{\delta}_t$ is the solution of the following equation:

$$\overline{\delta}_t = \frac{R_{t+1}}{\omega\left(\overline{\delta}_t, z_t\right) sA}. \tag{22}$$

Note that those adult individuals with $\delta^i > \overline{\delta}_t$ satisfy $q_{t+1}^i > R_{t+1}/\omega_t^i$ and, hence, are entrepreneurs. The rest are savers. Therefore, the fraction of adult entrepreneurs in t, λ_t, satisfies $\lambda_t = 1 - F\left(\overline{\delta}_t\right)$.

The solution to equation (22) is a function $\overline{\delta}_t = \widetilde{\delta}\left(R_{t+1}, z_t\right)$. Using (21) and Assumption A, we obtain that this function is increasing in R_{t+1} and decreasing in the ratio z_t when $\xi > 0$. Since the fraction of entrepreneurs decreases with $\overline{\delta}_t$, we obtain that this fraction decreases with R_{t+1} and increases with z_t. The intuition is as follows. A larger return of financial assets, R_{t+1}, decreases the number of entrepreneurs, since more individuals obtain a larger utility when they only invest in financial assets. In contrast, an increase in the ratio z_t makes adult individuals wealthier and, as a consequence, more individuals find affordable the start-up cost and become entrepreneurs. This positive effect of wealth on entrepreneurship is well-known and supported by empirical evidence (see, for instance, Quadrini, 2009).

3 Equilibrium

3.1 Intertemporal equilibrium

We determine the equilibrium using the market clearing conditions for productive capital and financial assets. The first one implies that the firms' aggregate demand of productive capital in period $t+1$, k_{t+1}, equals the aggregate supply of productive capital that is obtained from the aggregation of the product between investment productivity, δ^i, and investments, κ_{t+1}^i, of each entrepreneur. Therefore, the market clearing condition in period $t+1$ is

$$k_{t+1} = \int_{\overline{\delta}_t}^{\delta_{\max}} \delta^i \kappa_{t+1}^i N f\left(\delta^i\right) d\delta^i, \tag{23}$$

where $f\left(\delta^i\right)$ is the density function of the distribution of abilities. We rewrite the market clearing condition for productive capital as

$$k_{t+1} = \gamma N w_t \tau_t, \tag{24}$$

where $\tau_t = \int_{\overline{\delta}_t}^{\delta_{\max}} \left(\delta^i \kappa_{t+1}^i / \gamma w_t\right) f\left(\delta^i\right) d\delta^i$. Using (19) and after some computations, we obtain

$$\tau_t \equiv \widetilde{\tau}\left(\overline{\delta}_t, R_{t+1}, z_t\right) = \int_{\overline{\delta}_t}^{\delta_{\max}} \frac{\delta^i \left[z_t \left(\delta^i\right)^{v_1} + (1-\xi)\phi\left(\delta^i\right)^{v_2}\right]}{1-\theta s A \delta^i / R_{t+1}} f\left(\delta^i\right) d\delta^i. \tag{25}$$

Equation (24) indicates that capital depends on the product between the wage per efficiency unit and τ_t, which is a measure of the aggregate efficiency units of labor of entrepreneurs that takes into account the productivity of investment and the investment multiplier. The term of τ_t inside the square brackets amounts for the efficiency units of labor at young and adult ages. Young individuals efficiency units are multiplied by the ratio z_t, because young individuals obtain labor income one period before individuals invest in capital, and adult individuals efficiency units are multiplied by $1-\xi$ to subtract the start-up cost.

From (25), it is immediate to see that τ_t decreases with $\overline{\delta}_t$ and R_{t+1} and increases with z_t. The intuition is quite immediate. First, an increase in $\overline{\delta}_t$ reduces the number of entrepreneurs and, as a result, capital accumulation decreases. Second, an increase in R_{t+1} reduces the amount of credit that can be obtained using capital as collateral. As a consequence, it reduces the investment multiplier when $\theta > 0$, which also reduces capital accumulation. Finally, adult individuals are wealthier when z_t increases, which explains the positive effect of this ratio on capital accumulation.

It is convenient to rewrite (24) and (25) in terms of R_{t+1} and z_t. To this end, we use the definition of a_t to obtain that $a_t = k_t / l_t$, where l_t is the total efficiency units of employment that satisfy $l_t = N\left(\chi^1 + \phi\chi^2\right)$, where

$$\chi^j = \int_{\delta_{\min}}^{\delta_{\max}} \left(\delta^i\right)^{v_j} f\left(\delta^i\right) d\delta^i, \; j = 1, 2.$$

We use the definitions of a_t and z_t and equation (3) to rewrite the market clearing condition for productive capital, (24), as

$$R_{t+1} = \frac{A\gamma(1-s)}{\chi^1 + \phi\chi^2} z_{t+1}\tau_t. \tag{26}$$

We next obtain the market clearing condition for financial assets. To this end, we obtain the aggregate value of financial assets. We first use (3), (13) and (14) to deduce that the aggregate value of the financial assets owned by young and adult savers is

$$x_{1t}^S = (1-s)ANa_t\left((\beta+\gamma)\eta_{t+1}^1 - \frac{\alpha\phi}{z_{t+1}}\eta_{t+1}^2\right), \tag{27}$$

$$x_{2t+1}^S = \gamma(1-s)ANa_{t+1}\left(z_{t+1}\eta_{t+1}^1 + \phi\eta_{t+1}^2\right), \tag{28}$$

where

$$\eta_{t+1}^j \equiv \widetilde{\eta}^j\left(\overline{\delta}_{t+1}\right) = \int_{\delta_{\min}}^{\overline{\delta}_{t+1}} \left(\delta^i\right)^{v_j} f\left(\delta^i\right) d\delta^i,\ j = 1, 2,$$

measures the aggregate efficiency units of labor of savers, which are an increasing function of $\overline{\delta}_{t+1}$.

Using (3), (17) and (18), we obtain that the aggregate value of the financial assets owned by young and adult entrepreneurs is

$$x_{1t}^E = (1-s)ANa_t\left((\beta+\gamma)\pi_{t+1}^1 - \alpha\frac{(1-\xi)\phi\pi_{t+1}^2}{z_{t+1}}\right), \tag{29}$$

$$x_{2t+1}^E = -\theta\gamma(1-s)ANsA\frac{a_{t+1}\tau_{t+1}}{R_{t+2}}, \tag{30}$$

where

$$\pi_{t+1}^j \equiv \widetilde{\pi}^j\left(\overline{\delta}_{t+1}\right) = \int_{\overline{\delta}_{t+1}}^{\delta_{\max}} \left(\delta^i\right)^{v_j} f\left(\delta^i\right) d\delta^i,\ j = 1, 2,$$

measures the aggregate efficiency units of labor of entrepreneurs, which are a decreasing function of $\overline{\delta}_{t+1}$. Observe that $\widetilde{\pi}^j\left(\overline{\delta}_{t+1}\right) + \widetilde{\eta}^j\left(\overline{\delta}_{t+1}\right) = \chi^j$.

We define the aggregate value of financial assets owned by individuals at period t as $\Psi_t = x_{1t}^E + x_{1t}^S + x_{2t}^E + x_{2t}^S$. Using (27)-(30), we obtain

$$\Psi_t = (1-s)ANa_t\Delta_t,$$

where

$$\Delta_t = (\beta+\gamma)\chi^1 - \frac{\alpha\phi\chi^2}{z_{t+1}} + \frac{\alpha\xi\phi\pi_{t+1}^2}{z_{t+1}} + \gamma\left(z_t\eta_t^1 + \phi\eta_t^2\right) - \frac{\theta\gamma sA\tau_t}{R_{t+1}}. \tag{31}$$

The market clearing condition for financial assets depends on the type of financial asset. In a bubbleless equilibrium, $b^i_{1,t} = b^i_{2,t} = 0$ and the financial assets are only deposits and credits. Since the aggregate value of deposits equals the aggregate value of credits in every period, the market clearing condition implies that the aggregate value of the financial assets owned by individuals is zero in every period; that is $\Psi_t = \int_{\delta_{\min}}^{\delta_{\max}} d^i_{1,t} N f\left(\delta^i\right) d\delta^i + \int_{\delta_{\min}}^{\delta_{\max}} d^i_{2,t} N f\left(\delta^i\right) d\delta^i = 0$ or, equivalently, $\Delta_t = 0$. In contrast, in a bubbly equilibrium, financial assets include the bubble and also deposits and credits. The equality between the aggregate values of deposits and credits implies that the aggregate value of the financial assets equals the value of the bubble, which is positive in a bubbly equilibrium; that is $\Psi_t = \int_{\delta_{\min}}^{\delta_{\max}} b^i_{1,t} N f\left(\delta^i\right) d\delta^i + \int_{\delta_{\min}}^{\delta_{\max}} b^i_{2,t} N f\left(\delta^i\right) d\delta^i > 0$. We assume that the supply of speculative assets is fixed. In this case, the market clearing condition states that the value of the bubble purchased at $t+1$ by young and adult individuals equals the value of the bubble that in period $t+1$ adult and old individuals sell. These individuals sell the bubble purchased in the previous period, Ψ_t, multiplied by the growth of the price, R_{t+1}. Therefore, the market clearing condition is

$$\Psi_{t+1} = R_{t+1} \Psi_t. \tag{32}$$

Using the definition of the ratio z_t, this market clearing condition can be rewritten as

$$\Delta_{t+1} = z_{t+1} \Delta_t. \tag{33}$$

The aggregate value of financial assets shows the difference between the two sources of external financing. When there is no bubble, the aggregate value of credit limits the amount of savings through financial assets, since $\Psi_t = 0$. In other words, the supply of savings is limited by the demand of savings. The existence of a financial bubble overcomes this limitation, since next generation purchases of the bubble, Ψ_{t+1}, are an additional source of demand of assets that provides liquidities in equilibrium. To gain more intuition, we can provide an interpretation of (32) in terms of demand and supply of financial assets. Ψ_t can be interpreted as the net supply of assets and it is increasing in R_{t+1}, as follows from (31). Following this interpretation, the net demand of financial assets is Ψ_{t+1}/R_{t+1} when there is a bubble and zero otherwise.

The larger demand of financial assets implies a larger return of these assets in the equilibrium with bubbles. We prove this result in the following section.

We use below the two market clearing conditions to define an equilibrium.

Definition 1 *An equilibrium of this economy is a path of* $\{R_t, z_t, \tau_t, \overline{\delta}_t, \Delta_t\}_{t=1}^{\infty}$ *that, given* z_1*, solves the two market clearing conditions (26) and (33), satisfies (22), (25) and (31), and along which the value of the bubble is non-negative,* $\Delta_t \geq 0$.

At this point, we introduce constraints on the domain of the distribution in order to ensure that the equilibrium is well-defined with a positive number of both savers and entrepreneurs. In particular, we assume that $\delta_{\max} < R_{t+1}/(\theta s A)$ so that the multiplier of investment is positive and finite for all individuals and we also assume that $\overline{\delta}_t \in (\delta_{\min}, \delta_{\max})$ so that $\lambda_t \in (0, 1)$. These assumptions are rewritten as constraints on the return of the financial assets in the following assumption:

Assumption B. $R_{t+1} > \max\{\omega(\delta_{min}, z_t) sA\delta_{\min}, \theta sA\delta_{\max}\} = \theta sA\delta_{\max}$ with $\omega(\delta_{min}, z_t)\delta_{\min} < \theta\delta_{\max}$ and $R_{t+1} < \omega(\delta_{\max}, z_t) sA\delta_{\max}$.

In the following section, we show that the equilibrium can converge to two different steady states: a bubbly steady state with $\Delta_t > 0$ and a bubbleless one with $\Delta_t = 0$. These steady states satisfy Assumption B.

3.2 Steady states

We denote by g^*, R^*, $\overline{\delta}^*$ and z^* the constant growth factor, interest factor, ability of the marginal individual and ratio z at the bubbly steady state and we denote by g^o, R^o, $\overline{\delta}^o$ and z^o the corresponding values of these variables at the bubbleless steady state. In this subsection, we obtain conditions that ensure existence and uniqueness of these two steady states. To this end, we rewrite the two market clearing conditions when $R_t = R$ and $z_t = z$ for all t as two functions relating z with R. The following proposition characterizes the function that describes the market clearing condition for productive capital.

Proposition 1 *The pairs* R *and* z *for which the market for productive capital clears satisfy the following increasing and continuous function:* $z = \varphi(R)$. *This function is* *defined in the domain* $R \in (\theta sA\delta_{\max}, \omega(\delta_{\max}, z)\, sA\delta_{\max})$ *and satisfies* $\varphi(\theta sA\delta_{\max}) = 0$ *and* $\varphi(\omega(\delta_{\max}, z)\, sA\delta_{\max})$ *diverges to infinite.*

Proof. See the online appendix. ■

We next use (22), (26) and $\pi^2 = \chi^2 - \eta^2$ to rewrite the market clearing condition for financial assets, (33), as

$$\Delta(1-z) = 0 \tag{34}$$

with $\Delta = \Omega(R,z) - Q$, where

$$\Omega(R,z) \equiv \gamma z\left[z\eta^1(R,z) + \phi\eta^2(R,z)\right] + (\beta+\gamma)\chi^1 z - \alpha\xi\phi\eta^2(R,z), \tag{35}$$

$$Q = \frac{\theta s\left(\chi^1 + \phi\chi^2\right)}{(1-s)} + \alpha\phi(1-\xi)\chi^2,$$

and $\eta^j(R,z) = \widetilde{\eta}^j\left(\widetilde{\delta}(R,z)\right)$ for $j = 1,2$.

We proceed to show the existence and uniqueness of the bubbly steady state. Given that at this steady state $\Delta > 0$, we deduce, from (34), that $z^* = 1$, which implies that $R^* = g^*$. Therefore, the growth of the bubble equals the growth of wages, which is a well-known result since Tirole (1985) and Grossman and Yanagawa (1993).[13] We next use the market clearing condition for productive capital to obtain that R^* is such that $1 = \varphi(R^*)$. Since $\varphi(R)$ is a continuous and increasing function, $\varphi(\theta sA\delta_{\max}) = 0$ and $\varphi(\omega(\delta_{\max}, z)\, sA\delta_{\max})$ diverges to infinite, there exists a unique R^* that clears the market of productive capital and satisfies Assumption B. $R_t = R^*$ and $z_t = 1$ define a bubbly steady state when the aggregate value of the speculative assets owned by individuals is positive; i.e. $\Delta > 0$ when $R_t = R^*$ and $z_t = 1$, which occurs when:

$$\Omega(R^*, 1) > Q. \tag{36}$$

We conclude that there exists a unique bubbly steady state when (36) is satisfied.

We next study the steady state without bubbles for which $\Delta = 0$ and, hence, $\Omega(R,z) = Q$. This equation implicitly defines a continuous function $z = \zeta(R)$, along which the value of financial assets is zero. The steady state without bubbles is the

[13] If $R^* > g^*$ the bubble is not sustainable because the price grows faster than the wage and if $R^* < g^*$ the bubble asymptotically vanishes, implying that the equilibrium converges to the bubbleless steady state.

solution to equations $z = \zeta(R)$ and $z = \varphi(R)$. We use these two functions to study existence and uniqueness of this steady state. The following proposition summarizes the results on the existence and uniqueness of the different types of steady state.

Proposition 2 *If Assumptions A and B and condition (36) are satisfied then:*

(i) There exists a unique bubbly steady state that satisfies $R^* = g^*$.

(ii) There exists a unique bubbleless steady state when ξ *is sufficiently small. This steady state satisfies* $R^o < g^o$ *and* $R^* > R^o$.

Proof. Results for the bubbly steady state follow from the previous arguments. Existence and uniqueness of the bubbleless steady state are shown in the online appendix. ■

While the existence of a steady state without bubbles does not depend on the shape of the function $\zeta(R)$, uniqueness does. In the online appendix, we show that the shape of the function $\zeta(R)$ depends on the value of the start-up cost rate. The function $\zeta(R)$ is downward slopping when ξ is small, which ensures uniqueness of the bubbleless steady state because the function $\varphi(R)$ is upward slopping.

In Proposition 2, we also compare the two steady states when we assume that ξ is small enough so that there is a unique bubbleless steady state. In the online appendix, we show that a small value ξ also implies that $\Omega(R, z)$ increases with z. Since $\Omega(R, z)$ is increasing in z, the value of the financial assets is positive when $z > \zeta(R)$ and zero when $z = \zeta(R)$. As a result, at the bubbly steady state $1 > \zeta(R^*)$ and at the bubbleless steady state $z^o = \zeta(R^o)$. These conditions and $\varphi(R)$ being increasing imply that $R^* > R^o$ and $z^o < 1$ when there is a unique bubbleless steady state. This comparison between steady states is shown graphically in Figure 2 of the online appendix.

The results in Proposition 2 imply that the bubble increases both R and z. On the one hand, the larger return of financial assets in the bubbly steady state is a well-known result that is explained because the bubble increases the savings devoted to the demand of financial assets (see, for instance, Farhi and Tirole, 2012). On the other hand, the smaller value of z at the bubbleless steady state implies that $R^o < g^o$. This is also a well-known relation that the bubbleless steady state must satisfy to ensure the existence of a bubbly steady state (see Grossman and Yanagawa, 1993).

In the online appendix, we study stability and show that when $\theta = 0$ and ξ is sufficiently small, the bubbleless steady state is locally stable and the bubbly steady state is either locally stable or saddle path stable. These results are obtained for restrictive parameter conditions. However, using numerical examples, we show that local stability of the bubbleless steady state and saddle path stability of the bubbly steady state seem a robust finding. These results imply that the dynamic path can converge to both steady states and, therefore, the particular steady state toward which the economy converges depends on individuals' expectations.

4 Entrepreneurs, growth and productivity

In this section, we study the effect of the bubble on the number of entrepreneurs, on economic growth and on TFP. In the first subsection, we describe the mechanisms through which the bubble affects these variables. In the second subsection, we use numerical examples to measure the effect of the bubble. Finally, we compare our findings with those of the related literature.

4.1 Extensive and intensive margins of the bubble

We first study the effect of the bubble on the composition of the population between entrepreneurs and savers. This composition effect is determined by the ability of the marginal individual, $\overline{\delta}_t$. As follows from (22), this ability increases with R_{t+1} and decreases with the ratio z_t. Since the bubble increases both R_{t+1} and z_t, the composition effect of the bubble is ambiguous. This ambiguity is the consequence of two opposite mechanisms. On the one hand, in a bubbly steady state the return on financial assets is larger, which implies that more individuals choose to be savers. On the other hand, adult individuals are wealthier with the bubble. As a result, more individuals find affordable the start-up cost. Therefore, the number of entrepreneurs is larger at the bubbly steady state when this cost mechanism dominates, which occurs when the start-up cost rate is sufficiently large.

The bubble increases the number of entrepreneurs by reducing the ability of the marginal individual. This implies that new entrepreneurs are less productive than existing ones. Moreover, these new entrepreneurs benefit from a smaller investment multiplier and a lower labor income and, hence, they invest less. Therefore, new entrepreneurs are less productive and smaller, which are two features observed in firm-level data.

We next analyze the effect of the bubble on growth. To this end, we use (26) and the definition of z_t to obtain

$$g_{t+1} = \frac{A\gamma\,(1-s)}{\chi^1 + \phi\chi^2}\widetilde{\tau}\left(\overline{\delta}_t, z_t, R_{t+1}\right). \tag{37}$$

Equation (37) shows that the bubble may only increase growth if it enlarges τ_t. In Section 3.1, we have shown that τ_t decreases with $\overline{\delta}_t$ and R_{t+1} and increases with z_t. Each of these three variables introduces a distinct effect of the bubble. The first variable, $\overline{\delta}_t$, measures the composition effect of the bubble. A larger $\overline{\delta}_t$ reduces the number of entrepreneurs and, as a consequence, capital accumulation and growth decrease. Since the bubble may either increase or decrease the number of entrepreneurs, the composition effect of the bubble on growth is ambiguous. It is positive when the start-up cost rate is large, since the number of entrepreneurs increases in this case. This effect is the extensive margin of the bubble and it is consistent with findings in the literature showing that more entrepreneurs increase growth (see Quadrini, 2009).

The second variable, z_t, measures the liquidity effect of the bubble, which has been introduced in other papers (Martin and Ventura, 2012, or Farhi and Tirole, 2012). Since part of the labor income is obtained in the first period of life, but investment can only be done in the second, the bubble provides liquidities that increase adult's wealth. As a consequence, adult individuals increase capital accumulation and growth.

To gain some additional intuition on the liquidity effect, we assume that $\theta = 0$. In this case, adult individuals save, as the aggregate financial assets of adult savers and entrepreneurs satisfy $x_{2,t}^S > 0$ and $x_{2,t}^E = 0$. Since in the bubbleless steady state the value of the aggregate financial assets equals zero, the aggregate financial assets of young individuals must be negative, i.e. $x_{1,t}^S + x_{1,t}^E < 0$. In other words, young individuals borrow from the deposits accumulated by adult savers. These loans are paid back when adult, which limits productive investment. In contrast, in the bubbly steady state, aggregate financial assets are positive. That is, young individuals can hold the bubble and sell it in the following period, even if adult individuals also buy the bubble to postpone consumption. As a consequence, the amount borrowed when young and the amount adult individuals must pay for the credit decline with the bubble. Adult individuals then are wealthier and can invest more in productive capital. In this way, the bubble provides liquidities to adult credit constrained entrepreneurs. Of course, the same intuition still applies when $\theta > 0$ but not too large.

We have seen that the bubble makes adult individuals wealthier through the liquidity effect. As a result, entrepreneurs increase investment and more individuals find affordable the start-up cost and become entrepreneurs. Thus, the aforementioned cost mechanism of the bubble can be interpreted as the extensive margin of the liquidity effect. Following this interpretation, the increase in investment of each entrepreneur corresponds to the intensive margin.

The last variable, R_{t+1}, measures the leverage effect of the bubble that has also been considered by other papers in the literature (Farhi and Tirole, 2012). By increasing R_{t+1}, the bubble reduces the amount of credit that can be obtained using productive investment as collateral. As a result, the investment multiplier of each entrepreneur decreases, which reduces aggregate investment and growth. From the expression of the investment multiplier, it is immediate to see that the leverage effect of the bubble on growth weakens when the degree of pledgeability, measured by θ, is smaller and disappears when $\theta = 0$.[14]

We conclude that it is more likely that the bubble increases growth when the degree of pledgeability is small and the start-up cost rate is large.

Finally, we consider the effect of the bubble on TFP. To obtain TFP, we use (5) and (23) to obtain $y_{t+1} = TFP_{t+1} \int_{\bar{\delta}_t}^{\delta_{\max}} \kappa_{t+1}^i N f\left(\delta^i\right) d\delta^i$, where

$$TFP_{t+1} = A \frac{\int_{\bar{\delta}_t}^{\delta_{\max}} \delta^i \kappa_{t+1}^i f\left(\delta^i\right) d\delta^i}{\int_{\bar{\delta}_t}^{\delta_{\max}} \kappa_{t+1}^i f\left(\delta^i\right) d\delta^i}.$$

[14] These results, related to the degree of pledgeability, are in line with those obtained in Clain-Chamosset-Yvrad et al. (2020). In this paper, we consider a model in which individuals of the same generation are identical and study how the growth effects of the bubble depend on the degree of pledgeability. We show that growth is larger with the bubble when the degree of pledgeability is sufficiently small.

Notice that TFP equals the average return that entrepreneurs obtain per unit invested. It depends on two margins: extensive, related to the number of entrepreneurs, and intensive, which is determined by the amounts invested by each entrepreneur. In what follows, we show how these two margins are affected by the bubble. To this end, we use (3), (19), and (25) to rewrite TFP as

$$TFP_{t+1} = A\frac{\tau_t\left(\overline{\delta}_t, R_{t+1}, z_t\right)}{\nu_t\left(\overline{\delta}_t, R_{t+1}, z_t\right)}, \tag{38}$$

where

$$\nu_t = \int_{\overline{\delta}_t}^{\delta_{\max}} \frac{1}{1-\theta q^i_{t+1}/R_{t+1}}\left[z_t\left(\delta^i\right)^{v_1} + (1-\xi)\,\phi\left(\delta^i\right)^{v_2}\right] f\left(\delta^i\right) d\delta^i.$$

TFP depends on the same three variables, $\overline{\delta}_t$, R_{t+1} and z_t, which measure the composition, leverage and liquidity effects of the bubble. The composition effect is the extensive margin of the bubble. An increase in $\overline{\delta}_t$ reduces the number of entrepreneurs and, since the remaining ones are more productive, TFP increases. Following the same argument, we assert that the composition effect reduces TFP when the bubble increases the number of entrepreneurs.

The leverage and liquidity effects of the bubble affect TFP through the intensive margin. The effect of this margin on TFP has been studied by Miao and Wang (2012) and Hirano and Yanagawa (2017). These authors show that the bubble increases TFP when it increases to a larger extend the investment of more productive entrepreneurs. This mechanism also explains how the leverage and liquidity effects modify TFP in this paper. First, the leverage effect of the bubble is the consequence of the reduction in the investment multiplier due to the increase in the returns of financial assets. This effect is larger for more productive entrepreneurs, as follows immediately from the expression of the multiplier. This implies that the leverage effect reduces to a larger extend the investment of more productive entrepreneurs and, as a result, it reduces TFP.

Second, the liquidity effect is the consequence of the increase in the wealth of entrepreneurs due to the increase in z_t. In the online appendix, we show that the liquidity effect increases TFP only when $v_1 > v_2$. In this case, the distribution of wages of the young individuals is more correlated with investment productivity than the distribution of wages of the adult. To understand how the liquidity effect affects TFP, remember that z_t determines the effect that the wages of the young individuals have on the wealth of adult individuals. Therefore, the liquidity effect of the bubble causes a larger increase in investment when the wages of the young individuals are larger. As a result, when v_1 is large and the wages of the young individuals are highly correlated with the productivity of investment, the bubble increases to a larger extend the investment of more productive entrepreneurs. In contrast, when v_2 is large, more productive entrepreneurs obtain larger wages when adult. In this case, the most productive entrepreneurs need less of the liquidity provided by the bubble to invest. This explains that the liquidity effect increases TFP when $v_1 > v_2$.

To summarize, we show that when the start-up cost rate is large, the degree of pledgeability is small and the productivity of investment is more correlated with the wages of the young than with the wages of the adult, the bubble increases the number of entrepreneurs, growth and productivity. We conclude that this model explains the different facts mentioned in the introduction.

4.2 Numerical examples

In this section, we use numerical examples to show that the bubble can increase growth, TFP and the number of entrepreneurs. Although the model is too simple to perform a serious quantitative exercise and the goal of these examples is illustrative only, we introduce some discipline and we set the parameters so that the bubbly steady state of the benchmark economy matches several targets of the US economy in the period 2010-2020. Table 1 summarizes the parameters of the calibration.

[Insert Table 1]

In the numerical examples, we first assume that abilities follow a Pareto truncated distribution with density function:

$$f(\delta) = \frac{\sigma \delta_{\min}^{\sigma}}{1 - (\delta_{\min}/\delta_{\max})^{\sigma}} \delta^{-(1+\sigma)},$$

where the parameter $\sigma > 0$ determines the shape of the density function. Second, technological parameters, s and A, are set so that the labor income share equals 60%.

Third, preference parameters, α, β and γ, are set so that the ratio of consumption expenditure to income equals 84% in the first period of life and to 85% in the second period. Fourth, the parameters of the density function, $\delta_{\min}$, $\delta_{\max}$ and σ, are set to have in the bubbly steady state an annual growth rate close to 2.5% and a fraction of the entrepreneurs in the labor force of 10.6%. Fifth, the labor earnings parameters ϕ and v_2 are set so that the labor income in the second period of life is 12% larger than in the first period and to match that the average labor income of the third quartile is 97% larger than the average labor income of the first quartile. Notice that the model does not generate the large inequalities observed in labor income. Finally, the rest of parameters of the benchmark economy are set so that the equilibrium of this economy is consistent with the facts explained in the introduction, which requires a low pledgeability of capital, $v_1 > v_2$ and a large start-up cost rate. In particular, we assume that productive capital cannot be used as collateral, $\theta = 0$, that $v_1 = 1$ and we set $\xi = 0.75$, which implies that the start-up cost is 17.45% of the GDP per capita.[15]

Table 2 shows the values of the return of financial assets, growth rate, fraction of entrepreneurs and productivity in the bubbly and in the bubbleless steady states. From the comparison between the two steady states of the benchmark economy, it follows that growth and the return of financial assets are clearly larger with bubbles and both the number of entrepreneurs and TFP are slightly larger in the bubbly steady state. Thus, in the benchmark economy, bubbles increase the number of entrepreneurs, growth and TFP, which is consistent with the empirical findings mentioned in the introduction. In addition, the large change in the growth rate compared to the small changes in TFP and number of entrepreneurs is also consistent with evidence. For instance, the US economy between 2007 and 2009 suffers a reduction in GDP growth, TFP and number of firms of 94%, 0.54% and 3%, respectively.[16] The benchmark economy generates a change between the two steady states of 72%, 0.43% and 0.93% in these same variables. Clearly, the change in the variables is of similar magnitude.

[15]Start-up cost is measured as a bureaucratic cost by Klapper et al. (2006). They report huge differences between countries ranging from 0.5% to 81% of per capita GDP.

[16]These growth rates are obtained using the variables defined in Figure 1.

[Insert Table 2]

In Table 2, we compare the benchmark economy with three different counterfactual economies to show the importance of the composition effect or extensive margin of the bubble. First, Economy 1 differs from the benchmark economy in the value of the start-up cost rate, which is substantially smaller. As a result, compared to the benchmark economy, there are more entrepreneurs at both steady states. This larger amount of entrepreneurs explains the larger growth and the smaller TFP in both steady states. Moreover, the reduction in the start-up cost rate weakens the cost mechanism, which explains that in Economy 1 there are more entrepreneurs in the bubbleless steady state than in the bubbly one. As a consequence, the composition effect of the bubble reduces the differences between the growth rates of the bubbly and the bubbleless steady states, whereas it increases substantially the differences in TFP. In Economy 2, the start-up cost rate is larger than in the benchmark economy. This larger cost rate explains the smaller fraction of entrepreneurs, which reduces growth and increases TFP in both steady states. In this economy, the cost mechanism is more intense than in the benchmark and, as a consequence, the fraction of entrepreneurs is substantially larger in the bubbly steady state than in the bubbleless one. This implies that the composition effect of the bubble is large and, as a result, TFP is smaller in the bubbly steady state. Finally, in Economy 3, the correlation between productivity of investment and wages is equal at both periods of life. As a consequence, the liquidity effect of the bubble does not affect TFP. Since the number of entrepreneurs is larger in the bubbly steady state, the TFP is smaller due to the composition effect. These numerical examples show that the composition effect may be sizeable and determine the overall effect that bubbles have on growth and productivity.

4.3 Comparison with the literature

We next compare the findings in this paper with the results obtained by the literature that studies the effects of financial bubbles in models with heterogeneous individuals. To organize this comparison, we distinguish three groups of models: (i) models with two exogenous groups of individuals: savers and entrepreneurs; (ii) with two exogenous groups of entrepreneurs: high and low ability entrepreneurs; and (iii) with endogenous composition of the population between savers and entrepreneurs.

(i) Savers and entrepreneurs

Martin and Ventura (2012), Farhi and Tirole (2012) and Raurich and Seegmuller (2019) among many others have considered models in which the population is divided in two constant groups of individuals. The model of Section 2 can be adapted to this setting by assuming that the distribution function of δ is discrete and has the following properties: a constant fraction λ of individuals has a high ability, δ^H, and the rest, $1-\lambda$, has low ability, δ^L. Moreover, we assume that the support of the distribution satisfies $\delta^L < R_{t+2}/(\omega s A) < \delta^H$. This assumption implies that in equilibrium individuals with low ability will be savers and individuals with high ability will be entrepreneurs. Therefore, the fraction of entrepreneurs in the population is constant, it is equal to the parameter λ, and it is not affected by the bubble. In other words, the bubble does not generate the composition effect.

Using (38), we obtain that $TFP = A\delta^H$. Therefore, since all entrepreneurs are identical, TFP is constant and it is not affected by the bubble. Moreover, using (37), we obtain that the ratio between the growth rate in the bubbly and in the bubbleless steady states equals

$$\frac{g^*}{g^o} = \left(\frac{\left(\delta^H\right)^{v_1-v_2} + \phi\left(1-\xi\right)}{z^o\left(\delta^H\right)^{v_1-v_2} + \phi\left(1-\xi\right)}\right)\left(\frac{1-\theta\delta^H sA/R^o}{1-\theta\delta^H sA/R^*}\right).$$

The ratio of growth rates is equal to the product of two ratios. The first one measures the liquidity effect of the bubble and it is larger than one because $z^o < 1$. This effect is reinforced by a high value of $\left(\delta^H\right)^{v_1-v_2}$. The second one measures the leverage effect of the bubble and it is smaller than one since $R^* > R^o$. Therefore, in models that divide the population into two constant and homogeneous groups of individuals, the bubble increases growth when the liquidity effect dominates the leverage effect. However, these models do not explain that TFP and entrepreneurship increase with the bubble.

(ii) High and low ability entrepreneurs

Miao and Wang (2012) and Hirano and Yanagawa (2017) consider two constant groups of entrepreneurs with different ability to account for the effects that the bubble may have on TFP through the intensive margin. The model of Section 2 can also be adapted to this context. To this end, we assume again that the distribution function of δ is discrete and has the following properties: a constant fraction λ^L of individuals has low ability, δ^L, a constant fraction λ^M has a middle ability, δ^M, and the rest, λ^H, has high ability, δ^H. Obviously, $\lambda^L + \lambda^M + \lambda^H = 1$. We also assume that $\delta^L < R_{t+2}/(\omega s A) < \delta^M$, which implies that low ability individuals are savers and the rest, middle and high ability individuals, are entrepreneurs.

As occurs in models with identical entrepreneurs, the fraction of entrepreneurs in the population is constant, equal to $\lambda^M + \lambda^H$, and it is not affected by the bubble. The bubble also increases growth when the liquidity effect dominates the leverage effect of the bubble. However, since there are two different groups of entrepreneurs, the bubble affects TFP through the intensive margin. To see this, we use (38) to obtain that TFP equals

$$TFP_{t+1} = A \frac{\delta^M \frac{z_t\left(\delta^M\right)^{v_1} + (1-\xi)\phi\left(\delta^M\right)^{v_2}}{1-\theta\delta^M sA/R_{t+1}} \lambda^M + \delta^H \frac{z_t\left(\delta^H\right)^{v_1} + (1-\xi)\phi\left(\delta^H\right)^{v_2}}{1-\theta\delta^H sA/R_{t+1}} \lambda^H}{\frac{z_t\left(\delta^M\right)^{v_1} + (1-\xi)\phi\left(\delta^M\right)^{v_2}}{1-\theta\delta^M sA/R_{t+1}} \lambda^M + \frac{z_t\left(\delta^H\right)^{v_1} + (1-\xi)\phi\left(\delta^H\right)^{v_2}}{1-\theta\delta^H sA/R_{t+1}} \lambda^H}.$$

It is immediate to see that TFP decreases with R_{t+1} and increases with z_t if and only if $v_1 > v_2$. Therefore, the bubble affects TFP through the leverage and liquidity effects.

Models with two groups of entrepreneurs can explain that growth and productivity are larger with bubbles. However, they do not consider the composition effect of the bubble, which can be sizeable, according to the numerical examples.

(iii) Endogenous composition of the population

Kunieda and Shibata (2016) consider a model with a continuous distribution function of abilities to study the effect of the bubble on growth when the composition of the population between savers and entrepreneurs is endogenous. The model of Section 2 is adapted to their setting when $\theta = \xi = 0$ and $v_j = 0$, $j = 1, 2$.

A first difference with this paper is that they do not introduce the start-up cost. As a consequence, the number of entrepreneurs is smaller in the bubbly steady state. A second difference is that productive capital is not used as collateral and, hence, the bubble does not cause the leverage effect. Therefore, as follows from (25) and (37), the bubble causes two opposite effects on growth. First, the smaller number of entrepreneurs reduces capital accumulation and growth. This is the composition effect of the bubble. Second, the bubble still has a growth enhancing liquidity effect. Thus, the bubble promotes growth when the reduction in the number of entrepreneurs is not too large. Finally, TFP simplifies as follows

$$TFP_{t+1}\left(\overline{\delta}_t\right) = A\frac{\int_{\overline{\delta}_t}^{\delta_{\max}} \delta^i f\left(\delta^i\right) d\delta^i}{\int_{\overline{\delta}_t}^{\delta_{\max}} f\left(\delta^i\right) d\delta^i}.$$

Since $v_j = 0$, the liquidity effect of the bubble does not modify TFP. Therefore, TFP only depends on the composition effect. Given that in this economy the bubble reduces the number of entrepreneurs, TFP increases.

5 Concluding remarks

Entrepreneurship, growth and TFP increase when there is a financial bubble and decline during financial crises. We explain these facts as the result of a transition between two steady states: a steady state without bubbles, in which financial assets consists only of deposits, and another one with bubbles, in which financial assets also include a pure speculative asset.

We show that the aforementioned facts can be explained in an overlapping generations growth model populated by heterogenous individuals that live for three periods. We distinguish three sources of heterogeneity. First, individuals are heterogeneous in the return of productive investment. This heterogeneity separates individuals in two groups: savers and entrepreneurs. Savers only invest in financial assets, whereas entrepreneurs borrow from the savers to invest in productive capital. A novelty of this paper is the introduction of a start-up cost that entrepreneurs must pay. The bubble changes the composition of the population between savers and entrepreneurs. On the one hand, the bubble increases the return of financial assets, which increases the amount of savers. On the other hand, the bubble makes adult individuals wealthier.

As a result, more individuals are willing to pay the cost and become entrepreneurs. We show that the bubble increases the number of entrepreneurs when this cost mechanism dominates.

Second, workers are heterogeneous because individuals work both when young and adult. Since investment can only be done when adult and part of the labor income is obtained when young, the bubble provides liquidities to credit constrained entrepreneurs. This is the liquidity effect of the bubble that increases growth. However, the bubble also increases the return of financial assets, which causes a negative growth effect when productive capital is used as collateral. This is the leverage effect of the bubble. Finally, the bubble also increases growth by increasing the number of entrepreneurs. This is the composition effect of the bubble, which is a major contribution of this paper.

Third, wages are heterogenous among workers of the same generation. This is another novelty of this paper. We show that this heterogeneity is necessary to explain that TFP is larger with bubbles. In particular, we show that the liquidity effect of the bubble increases TFP when the productivity of investment is more correlated with the wages of the young than with the wages of the adult. In this case, the bubble further increases the investment of more productive entrepreneurs, which causes the increase in TFP.

We conclude that the model explains the aforementioned facts when we introduce two assumptions: a large start-up cost rate and a larger correlation between the productivity of investment and the wages in the first period of life than with the wages in the second period. The first assumption introduces a wealth effect, which is needed to explain that entrepreneurship is larger with the bubble. Therefore, the positive effect of bubbles on entrepreneurship does not depend on the particular functional form of the start-up cost, but on the introduction of a wealth effect.

The second assumption is needed to explain a larger TFP with the bubble. More precisely, this assumption ensures that more productive entrepreneurs obtain a larger income in the first period. Therefore, this second assumption could be generalized by assuming that the return of productive investment is correlated with income in the first period, which in addition to labor income could also include transfers, bequests or profits. This suggests that similar results could be obtained in different settings. As an example, we conjecture that the results in this paper could be obtained in models in which individuals decide in the first period between being workers and entrepreneurs. Individuals that decide to be entrepreneurs in the first period would also be entrepreneurs in the second period. Since more productive entrepreneurs obtain larger income (profits) in the first period of life, these more productive entrepreneurs would benefit more in the second period of the liquidities provided by the bubble. As a result, the bubble would further increase investment of more productive entrepreneurs. This example could be an alternative model that, using the same mechanisms, explains the positive effects of bubbles on TFP.

Acknowledgements

Xavier Raurich thanks financial support from the Spanish Government through grant RTI2018-093543-B-I00. Thomas Seegmuller thanks the financial support of the French National Research Agency Grant ANR-17-EURE-0020 and ANR-15-CE33-0001-01 and by the Excellence Initiative of Aix-Marseille University - A*MIDEX. This paper benefits from the comments of Anpeng Lee, of the participants of seminars at the Dongbei University, the University of Guanajuato, the University of Porto and of the participants in the Financial and Real Interdependencies 2021 workshop, at the Africa 2021 meeting of the Econometric Society, at the LAGV 2021 conference, at the SAET 2021 meeting and at the ASSET 2021 conference.

Supplementary material

An online appendix with supplementary material associated with this article is available at https://sites.google.com/view/xavier-raurich.

References

[1] Aghion, P., Bergeaud, A., Cette, G., Lecat, R. and Maghin H. (2019). Coase lecture. The Inverted-U relationship between credit access and productivity growth. *Economica*, 86, 1-31.

[2] Basu, S. and Fernald, J. (2001). Why is productivity procyclical? Why do we care?, in New Development in Productivity Analysis, eds. Charles R. Hulten, Edwin R. Dean, and Michael J. Harper, 224–296. NBER. Available at http://www.nber.org/chapters/c10128.

[3] Basu, S., Fernald, J. and Kimball, M. (2006). Are technology improvements contractionary?, *American Economic Review*, 96, 1418-1448.

[4] Bartelsman, E. and Doms, M. (2000). Understanding productivity: Lessons from longitudinal microdata. *Journal of Economic Literature*, 38 (3), 569-594.

[5] Basco, S. (2016). Switching bubbles: From outside to inside bubbles. *European Economic Review*, 87, 236-255.

[6] Bengui, J. and Phan, T. (2018). Asset pledgeability and endogenously leveraged bubbles. *Journal of Economic Theory*, 177, 280-314.

[7] Bilbiie, F., Ghironi, F. and Melitz, M. (2012). Endogenous entry, product variety, and business cycles. *Journal of Political Economy*, 120 (2), 304-345.

[8] Buera, F.J., Kaboski, J. P. and Shin, Y. (2011). Finance and development: A tale of two sectors. *American Economic Review,* 101, 1964-2002.

[9] Caballero, R. J., Farhi, E. and Hammour, M. L. (2006). Speculative growth: hints from the U.S. economy. *American Economic Review* 96, 1159-1192.

[10] Campbell, J. (1998). Entry, exit, embodied technology, and business cycles. *Review of Economic Dynamics*, 1, 371-408.

[11] Campbell, J. (1999). Asset prices, consumption, and the business cycle. Handbook of Macroeconomics, Volume 1, Part C, Chapter 19, 1231-1303.

[12] Chatterjee, S., Cooper, R. and Ravikumar, B. (1993). Strategic complementarity in business formation: aggregate fluctuations and sunspot equilibria, *Review of Economic Studies*, 60, 795-811.

[13] Clain-Chamosset-Yvrad, L. Raurich, X. and Seegmuller, T. (2020). Are the liquidity and collateral roles of asset bubbles different? AMSE WP 2020 - 10.

[14] Clementi, G. and Palazzo, B. (2016). Entry, exit, firm dynamics, and aggregate fluctuations. *American Economic Journal: Macroeconomics*, 8, 1-41.

[15] Cooley, T.F. and Quadrini, V. (2001). Financial markets and firm dynamics. *American Economic Review,* 91, 1286-1310.

[16] Fagereng, A., Guisso, L., Malacrino, D. and Pistaferri, L. (2020). Heterogeneity and persistence in returns to wealth. *Econometrica*, 88, 115-170.

[17] Farhi, E. and Tirole, J. (2012). Bubbly liquidity. *Review of Economic Studies,* 79, 678-706.

[18] Fernald, J. and Wang, C. (2016). Why has the cyclicality of productivity changed? What does it mean? *Annual Review of Economics*, 8, 465-496.

[19] Field, A. (2010). The procyclical behavior of total factor productivity in the United States, 1890-2004. *The Journal of Economic History*, 70, 326-350.

[20] Graczyk, A. and Phan, T. (2021). Regressive welfare effects of housing bubbles. *Macroeconomic Dynamics*, 25, 2102-2127.

[21] Grossman, G. and Yanagawa, N. (1993). Asset bubbles and endogenous growth. *Journal of Monetary Economics,* 31, 3-19.

[22] Hillebrand, M., Kikuchi, T. and Sakuragawa, M. (2018). Bubbles and crowding-in of capital via a savings glut. *Macroeconomic Dynamics*, 22, 1238-1266.

[23] Hirano, T. and N. Yanagawa (2017). Asset bubbles, endogenous growth, and financial frictions. *The Review of Economic Studies,* 84, 406-443.

[24] Jeong, H. and Townsend, R.M. (2007). Sources of TFP growth: occupational choice and financial deepening. *Economic Theory*, 32, 179-221.

[25] Klapper, L., Laeven, L. and Rajan, R. (2006). Entry regulation as a barrier to entrepreneurship. *Journal of Financial Economics,* 82, 591-629.

[26] Klapper, L., and Love, I. (2011). The impact of financial crisis in new firm registration, *Economics Letters*, 113, 1-4.

[27] Kocherlakota, N. (2009). Bursting bubbles: consequences and cures, mimeo, Federal Reserve Bank of Minneapolis.

[28] Koellinger, P. and Thurik, R. (2012). Entrepreneurship and the business cycle. *The Review of Economics and Statistics*, 94, 1143-1156.

[29] Kunieda, T. (2008). Asset bubbles and borrowing constraints. *Journal of Mathematical Economics*, 44, 112-131.

[30] Kunieda, T. (2014). A note on the crowd-in effect of asset bubbles in the perpetual youth model. *Mathematical Social Sciences*, 72, 50-54.

[31] Kunieda, T. and Shibata, A. (2016). Asset bubbles, economic growth, and a self-fulfilling financial crisis. *Journal of Monetary Economics*, 82, 70-84.

[32] Lee, Y. and Mukoyama, T. (2015). Entry and exit of manufacturing plants over the business cycle. *European Economic Review*, 77, 20-27.

[33] Martin, A. and Ventura, J. (2012). Economic growth with bubbles. *American Economic Review,* 102, 3033-58.

[34] Martin, A. and Ventura, J. (2016). Managing credit bubbles. *Journal of the European Economic Association,* 14, 753-789.

[35] Meza, F. and Quintin, E. (2005). Financial crises and total factor productivity. Center for Latin America Working Papers, Federal Reserve Bank of Dallas no. 105.

[36] Miao, J. and Wang P. (2012). Bubbles and total factor productivity. *American Economic Review: Papers & Proceedings*, 102, 82–87.

[37] Miao, J. and Wang P. (2018). Asset bubbles and credit constraints. *American Economic Review,* 108, 2590-2628.

[38] Midrigan, V. and Xu, D. Y. (2014). Finance and misallocation: evidence from plant-level data. *American Economic Review,* 104, 422-458.

[39] Moll, B. (2014). Productivity losses from financial frictions: can self-financing undo capital misallocation?. *American Economic Review,* 104, 3186-3221.

[40] Poschke, M. (2013). Who becomes an entrepreneur? Labor market prospects and occupational choice. *Journal of Economics Dynamics and Control*, 37, 693-710.

[41] Pratap, S. and Urrutia, C. 2012. Financial frictions and total factor productivity: accounting for the real effects of financial crises. *Review of Economic Dynamics*, 15, 336-358.

[42] Queralto, A. (2011). Financial market frictions, productivity growth and crises in emerging economies. Meeting Papers 697, Society for Economic Dynamics.

[43] Quadrini, V. (2009). Entrepreneurship in macroeconomics. *Annals of Finance,* 5, 295-311.

[44] Raurich, X. and Seegmuller, T. (2019). On the interplay between speculative bubbles and productive investment. *European Economic Review,* 111, 400-420.

[45] Tang, H. (2018). Asset price bubbles and the distribution of firms. Meeting Papers 362, Society for Economic Dynamics.

[46] Tian, C. (2018). Firm-level entry and exit dynamics over the business cycles. *European Economic Review*, 102, 298–326.

[47] Tirole, J. (1985). Asset bubbles and overlapping generations. *Econometrica,* 53, 1071-1100.

Tables

Table 1

Parameters	Values	Targets	Data	Model
A	10	Normalization	- -	- -
s	0.4	Labor Income Share[a]	60%	60%
α	0.422	Consumption to Inc. ratio of the young[b]	84%	84%
β	0.411	Consumption to Inc. ratio of the adult[b]	85%	85%
γ	0.167	Normalization	- -	- -
δ_{min}	1	Normalization	- -	- -
δ_{max}	16	Annual growth rate [a]	2.5%	2.6%
σ	1.214	Fraction of entrepreneurs in labor force[c]	10.6%	10.8%
ϕ	2.2	Adult to young labor Income[d]	1.12	1.15
v_2	0.4	Third to first labor income quartile[d]	1.97	1.52
ξ	0.75	Start-up cost to GDP	- -	17.45%
v_1	1	Assumption	- -	
θ	0	Assumption	- -	

Notes: [a] We obtain the labor income share and the annual growth rate from the PWT 9.1. [b] We obtain the consumption expenditure to income ratio from the 2019 US consumption expenditure survey. It is defined as the average value of the ratio between annual expenditures and after tax income for households whose reference person is aged between 25 and 44 years for the young and the reference person is aged between 45 and 64 for the adult. [c] The fraction of entrepreneurs is obtained from the OECD as the ratio between self-employed (both own-account workers and also self-employed who are employers) and total employment. This ratio changes substantially among OECD countries. In the US, this ratio is 10.6 per cent. [d] Data on the distribution of labor income across age groups and income quartiles is obtained from the US Bureau of Labor Statistics, first quartile of 2020. The income of the age groups is defined as the median income of individuals aged between 25 and 44 for the young and between 45 and 64 for the adult.

Table 2

Bubbly	Benchmark	Economy 1	Economy 2	Economy 3
$R = g$	1.70	2.30	1.20	0.44
λ	0.108	0.198	0.063	0.041
TFP	92.3	75.8	107	115
Value bubble	1.11	1.01	1.16	0.27
Bubbleless				
R	0.53	0.87	0.30	0.32
g	0.99	1.62	0.56	0.38
λ	0.107	0.267	0.047	0.039
TFP	91.9	67.1	114	117

Note. Economy 1: $\xi = 0.5$; Economy 2: $\xi = 1$; Economy 3: $v_1 = v_2 = 0.4$.

Online Appendix to Entrepreneurship, growth and productivity with bubbles

Lise Clain-Chamosset-Yvrard, Xavier Raurich and Thomas Seegmuller

A Individual's decisions

Individuals decide on consumption and investment to maximize the utility (10) subject to the budget constraints (6), (7) and (8), the borrowing constraint (9) and $\kappa^i_{2,t+1} \geq 0$. Given that the start-up cost does not depend on $\kappa^i_{2,t+1}$, there is a discontinuity in the Lagrangian function. As a consequence, we solve this maximization problem following a two-step procedure. First, we obtain the demands of consumption and assets of both savers and entrepreneurs from the first order conditions of the Lagrangian associated to the individuals' maximization problem. Second, we compare the indirect utility function of savers and entrepreneurs to determine the number of entrepreneurs.

Let $\lambda^i_{1,t}$, $\lambda^i_{2,t+1}$, $\lambda^i_{3,t+2}$, μ^i_{t+1} and η^i_{t+1} be, respectively, the Lagrangian multipliers of individual i associated to (6), (7), (8), (9) and $\kappa^i_{2,t+1} \geq 0$. Then, from the first order conditions with respect to $c^i_{1,t}$, $c^i_{2,t+1}$, and $c^i_{3,t+2}$, we obtain that $\lambda^i_{1,t} = \alpha / c^i_{1,t}$, $\lambda^i_{2,t+1} = \beta / \left(c^i_{2,t+1} - f^i_{t+1} \right)$, and $\lambda^i_{3,t+2} = \gamma / c^i_{3,t+2}$. Next, from the first order conditions with respect to d^i_{1t}, b^i_{1t}, d^i_{2t+1} and b^i_{2t+1}, we obtain that $R_{d,t+1} = R_{1,t+1} = R_{2,t+1} = R_{t+1}$ and

$$\lambda^i_{1,t} = \lambda^i_{2,t+1} R_{t+1}, \tag{39}$$

$$\lambda^i_{2,t+1} = \left(\lambda^i_{3,t+2} + \mu^i_{t+1} \right) R_{t+2}. \tag{40}$$

Finally, the first order condition with respect to investment implies

$$\lambda^i_{2,t+1} = q^i_{t+2} \lambda^i_{3,t+2} + \theta q^i_{t+2} \mu^i_{t+1} + \eta^i_{t+1}. \tag{41}$$

Given that entrepreneurs pay a cost, (41) solves the individuals' maximization problem only when $\kappa^i_{2,t+1} > 0$.

The solution of the individuals' maximization problem is characterized by the first order conditions (39), (40) and (41) together with the Kuhn-Tucker conditions

$$\mu_{t+1}^{i}\left[\theta q_{t+2}^{i}\kappa_{2,t+1}^{i}+R_{t+2}\left(b_{2t+1}^{i}+d_{2t+1}^{i}\right)\right]=0, \tag{42}$$

$$\eta_{t+1}^{i}\kappa_{2,t+1}^{i}=0. \tag{43}$$

We distinguish among four possible solutions. First, if $\eta_{t+1}^{i}>0$ and $\mu_{t+1}^{i}>0$ then (42) and (43) imply $\kappa_{2,t+1}^{i}=0$ and $b_{2t+1}^{i}+d_{2t+1}^{i}=0$. The budget constraint (8) implies that $c_{3,t+2}^{i}=0$, which is not a solution of the individuals' maximization problem.

Second, if $\eta_{t+1}^{i}=0$ and $\mu_{t+1}^{i}=0$ then $\kappa_{2,t+1}^{i}>0$ and (40) and (41) imply that $q_{t+2}^{i}=R_{t+2}$, i.e. all assets are perfect substitutes. Because of the start-up cost, the utility obtained when an individual invests in capital will be strictly lower than the level of utility if she does not invest in productive capital. Therefore, $\kappa_{2,t+1}^{i}>0$ cannot be a solution of the individual's maximization problem in this second case.

Third, if $\eta_{t+1}^{i}>0$ and $\mu_{t+1}^{i}=0$ then (42) and (43) imply that $\kappa_{2,t+1}^{i}=0$ and (39) and (40) imply that $\lambda_{1,t}^{i}=\lambda_{2,t+1}^{i}R_{t+1}$ and $\lambda_{2,t+1}^{i}=\lambda_{3,t+2}^{i}R_{t+2}$. This case characterizes the savers that do not invest in productive capital and are not credit constrained. From the last two equations, we obtain that $c_{2,t+1}^{i}=\left(\beta/\alpha\right)c_{1,t}^{i}R_{t+1}$ and $c_{3,t+2}^{i}=\left(\gamma/\beta\right)c_{2,t+1}^{i}R_{t+2}$. Combining these two equations with the budget constraints (6)-(8), we obtain (13), (14) in the main text and the optimal consumptions are given by:

$$c_{1,t}^{i}=\left(\alpha/R_{t+1}\right)\left(R_{t+1}w_{y,t}^{i}+w_{a,t+1}^{i}\right), \tag{44}$$

$$c_{2,t+1}^{i}=\beta\left(R_{t+1}w_{y,t}^{i}+w_{a,t+1}^{i}\right), \tag{45}$$

$$c_{3,t+2}^{i}=\gamma R_{t+2}\left(R_{t+1}w_{y,t}^{i}+w_{a,t+1}^{i}\right). \tag{46}$$

Fourth, if $\eta_{t+1}^{i}=0$ and $\mu_{t+1}^{i}>0$ then (42) and (43) imply that $\kappa_{2,t+1}^{i}>0$ and $\theta q_{t+2}^{i}\kappa_{2,t+1}^{i}+R_{t+2}\left(b_{2t+1}^{i}+d_{2t+1}^{i}\right)=0$. Therefore, this case characterizes the entrepreneurs that are credit constrained. We combine (39), (40) and (41) to obtain $\lambda_{1,t}^{i}=\lambda_{2,t+1}^{i}R_{t+1}$ and $\lambda_{2,t+1}^{i}=\left(1-\theta\right)\psi_{t+2}^{i}q_{t+2}^{i}\lambda_{3,t+2}^{i}$, where $\psi_{t+2}^{i}=1/\left(1-\theta q_{t+2}^{i}/R_{t+2}\right)$. From these two equations and taking into account that entrepreneurs pay the start-up cost, we obtain that $c_{2,t+1}^{i}-f_{t+1}^{i}=\left(\beta/\alpha\right)c_{1,t}^{i}R_{t+1}$ and

$$c_{3,t+2}^{i}=\left(\gamma/\beta\right)\left(1-\theta\right)\psi_{t+2}^{i}q_{t+2}^{i}\left(c_{2,t+1}^{i}-f_{t+1}^{i}\right).$$

We note that $\psi^i_{t+2} > 0$ requires $R_{t+2} > \theta q^i_{t+2}$. Combining these last two equations, the binding credit constraint and the budget constraints (6)-(8), we obtain equations (17)-(19) in the main text and consumptions are given by

$$c^{i,E}_{1,t} = (\alpha/R_{t+1})\left(R_{t+1}w^i_{y,t} + w^i_{a,t+1} - f^i_{t+1}\right), \tag{47}$$

$$c^{i,E}_{2,t+1} = \beta\left(R_{t+1}w^i_{y,t} + w^i_{a,t+1} - f^i_{t+1}\right) + f^i_{t+1}, \tag{48}$$

$$c^{i,E}_{3,t+2} = \gamma(1-\theta)\psi^i_{t+2}q^i_{t+2}\left(R_{t+1}w^i_{y,t} + w^i_{a,t+1} - f^i_{t+1}\right), \tag{49}$$

From the previous analysis, we distinguish between two groups of individuals. First, savers that do not invest in productive capital and are not credit constrained. Second, entrepreneurs that invest in productive capital and are credit constrained. In what follows, we determine the fraction of each group of individuals in the population. Since the start-up cost introduces a discontinuity, this analysis cannot follow from the first order condition with respect to capital. Instead, we compare the level of utility that an individual that is adult in period t attains as an entrepreneur and as a saver. To this end, we obtain the indirect utility functions of both agents by substituting in the utility function, (10), the optimal consumption demands of savers, (44)-(46), and of entrepreneurs, (47)-(49). We obtain that the indirect utility function when the individual is a saver is

$$u^{i,S}_{t-1} = \ln\left(w^i_t + R_t w^i_{t-1}\right) - \alpha \ln R_t + \gamma \ln R_{t+1} + \upsilon,$$

where

$$\upsilon = \ln(\alpha)^\alpha + \ln(\beta)^\beta + \ln(\gamma)^\gamma.$$

The indirect utility of the same individual when she is an entrepreneur is

$$u^{i,E}_{t-1} = \ln\left(w^i_t + R_t w^i_{t-1} - f^i_t\right) - \alpha \ln R_t + \gamma \ln\left[(1-\theta)\psi^i_{t+1}q^i_{t+1}\right] + \upsilon.$$

This individual decides to be an entrepreneur if $u^{i,E}_{t-1} \geq u^{i,S}_{t-1}$, which happens when $q^i_{t+1} \geq R_{t+1}/\omega^i_t$, where ω^i_t is defined in (20) in the main text. Remember that $q^i_{t+1} < R_{t+1}/\theta$. This inequality is compatible with $q^i_{t+1} \geq R_{t+1}/\omega^i_t$ because $\omega^i_t \in (\theta, 1)$ but implies an upper bound for q^i_{t+1}. This upper bound is introduced in Assumption B.

B Proof of Proposition 1

In this appendix, we characterize the function $z = \varphi(R)$ that relates all pairs R and z for which the market for productive capital clears. To this end, we first use (22) to rewrite $\tau = \widetilde{\tau}\left(R, \overline{\delta}, z\right)$ as $\tau = \tau(R, z)$. Remember that $\widetilde{\tau}\left(R, \overline{\delta}, z\right)$ is decreasing in both R and $\overline{\delta}$ and increasing in z. Moreover, (22) implies that $\overline{\delta}$ is an increasing function of R and a decreasing function of z. These relations imply that $\tau(R, z)$ is decreasing in R and increasing in z. We use $\tau(R, z)$ to rewrite (26) as

$$R = \frac{\gamma(1-s)A}{\chi^1 + \phi\chi^2} z\tau(R, z). \tag{50}$$

Equation (50) implicitly defines an increasing and continuous function, $z = \varphi(R)$, that provides all pairs of R and z for which the market of productive capital clears. As follows from Assumption B, this function is defined in the domain:

$$R \in \left(\theta s A \delta_{\max}, \omega(\delta_{\max}, z)\, sA\delta_{\max}\right).$$

To finish the proof of Proposition 1, we evaluate the function $\varphi(R)$ at the extremes of the domain. First, when R tends to $\theta sA\delta_{\max}$, both the investment multiplier associated to $\delta_{\max}$ and $\tau(\theta sA\delta_{\max}, z)$ diverge to infinite. Using (50), we deduce that $\varphi(\theta sA\delta_{\max}) = 0$. Second, when R tends to $\omega(\delta_{\max}, z)\, sA\delta_{\max}$, we obtain that $\overline{\delta} = \delta_{\max}$. As a result, the number of entrepreneurs equals zero and $\tau(\omega(\delta_{\max}, z)\, sA\delta_{\max}, z) = 0$. Using (50), we deduce that $\varphi(\omega(\delta_{\max}, z)\, sA\delta_{\max})$ diverges to infinite.

C Proof of Proposition 2

We next show the existence of a steady state without bubbles. To this end, we first use (26) and $\pi^2 = \chi^2 - \eta^2$ to obtain $\Delta = \widetilde{\Omega}\left(\overline{\delta}, z\right) - Q$, where

$$\widetilde{\Omega}\left(\overline{\delta}, z\right) \equiv \gamma z\left[z\widetilde{\eta}^1\left(\overline{\delta}\right) + \phi\widetilde{\eta}^2\left(\overline{\delta}\right)\right] + (\beta+\gamma)\chi^1 z - \alpha\xi\phi\widetilde{\eta}^2\left(\overline{\delta}\right). \tag{51}$$

We define the function $z = z^o\left(\overline{\delta}\right)$ as the solution to equation $\widetilde{\Omega}\left(\overline{\delta}, z\right) = Q$. This function provides the values of z and $\overline{\delta}$ for which $\Delta = 0$. We use this function to define $R_1 = sA\omega(\delta_{\min}, z^o(\delta_{\min}))\,\delta_{\min}$ and $R_2 = sA\omega(\delta_{max}, z^o(\delta_{\max}))\,\delta_{\max}$, which are, respectively, the lower and upper values of the range of R that are consistent with $\lambda_t \in (0,1)$. Assumption B and $\omega_t > \theta$ imply that $\theta\delta_{\max} sA \in (R_1, R_2)$.

Notice that $\eta^j (R_1, z) = 0$ and $\eta^j (R_2, z) = \chi^j$ for all $j = 1, 2$. Using these relations and $\Omega(R, z) = Q$, where $\Omega(R, z)$ is defined in (35), we obtain that $\zeta(R_1) > 0$ and $\zeta(R_2) < \infty$.

Using the properties of the function $\varphi(R)$, we obtain that $\zeta(R_1) > 0 = \varphi(\theta\delta_{\max} sA)$ and $\zeta(R_2) < \varphi(R_2) = \infty$. These conditions imply that the functions $\zeta(R)$ and $\varphi(R)$ cross, which guarantees the existence of at least one steady state without bubbles.

Uniqueness of the steady state without bubbles depends on the shape of the function $\zeta(R)$, which is implicitly obtained from $\Omega(R, z) = Q$. To obtain the slope of this function, we first introduce conditions that ensure that the function $\Omega(R, z)$ increases with z. The derivative of this function is:

$$\begin{aligned} \frac{\partial \Omega(R,z)}{\partial z} &= \gamma\left[2z\eta^1(R,z) + \phi\eta^2(R,z)\right] + (\beta+\gamma)\chi^1 + \\ &+ \left[\gamma z\left(z\overline{\delta}^{v_1} + \phi\overline{\delta}^{v_2}\right) - \alpha\phi\xi\overline{\delta}^{v_2}\right] f\left(\overline{\delta}\right)\frac{\partial\overline{\delta}}{\partial z}, \end{aligned}$$

and, using (22) in the main text, we obtain $\partial\overline{\delta}/\partial z = -\vartheta\xi < 0$ where

$$\vartheta = \frac{\frac{\overline{\delta}}{\gamma}\left(1 - \frac{\phi\xi(\overline{\delta})^{v_2}}{\phi(\overline{\delta})^{v_2} + (\overline{\delta})^{v_1} z}\right)^{\frac{1}{\gamma}-1}(1-\theta)\frac{\phi(\overline{\delta})^{v_2+v_1}}{[\phi(\overline{\delta})^{v_2} + (\overline{\delta})^{v_1} z]^2}}{\omega + \frac{\overline{\delta}}{\gamma}\left(1 - \frac{\phi\xi(\overline{\delta})^{v_2}}{\phi(\overline{\delta})^{v_2} + (\overline{\delta})^{v_1} z}\right)^{\frac{1}{\gamma}-1}(1-\theta)\frac{\xi\phi z(v_1 - v_2)(\overline{\delta})^{v_1+v_2-1}}{[\phi(\overline{\delta})^{v_2} + (\overline{\delta})^{v_1} z]^2}} > 0.$$

Therefore, we obtain that

$$\begin{aligned} \frac{\partial \Omega(R,z)}{\partial z} &= \gamma\left[2z\eta^1(R,z) + \phi\eta^2(R,z)\right] + (\beta+\gamma)\chi^1 \\ &- \left[\frac{\gamma z\left(z\overline{\delta}^{v_1 - v_2} + \phi\right)}{\alpha\phi} - \xi\right]\alpha\phi\overline{\delta}^{v_2} f\left(\overline{\delta}\right)\vartheta\xi. \end{aligned}$$

This derivative is positive when ξ is sufficiently small. We assume that ξ is small so that the derivative is positive. We next analyze the sign of the following derivative:

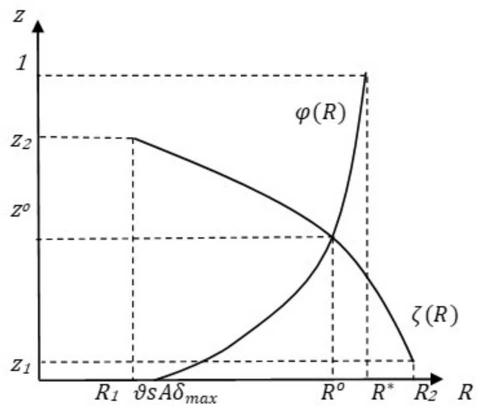
$$\frac{\partial \Omega(R,z)}{\partial R} = \left[\gamma z\left(z\overline{\delta}^{v_1} + \phi\overline{\delta}^{v_2}\right) - \alpha\phi\xi\overline{\delta}^{v_2}\right] f\left(\overline{\delta}\right)\frac{\partial\overline{\delta}}{\partial R}.$$

Since $\partial\overline{\delta}/\partial R > 0$, this derivative is positive when ξ is small.

Therefore, when ξ is small $\Omega(R, z)$ increases with both R and z, which implies that the function $z = \zeta(R)$ is decreasing. Since $\varphi(R)$ is increasing, uniqueness of the bubbleless steady state is ensured for small values of ξ.

Figure 2 plots the functions $z = \varphi(R)$ and $z = \zeta(R)$, when the latter is decreasing. This figure shows the existence of the two steady states. From the comparison between the two steady state, we observe that $R^* > R^o$ and $z^o < 1$.

Figure 2. Bubbly and bubbleless steady states



D Local stability

We assume that $\theta = 0$ to simplify the analysis of stability. Then, equations (22), (26), (31) and (33) in the main text that characterize the equilibrium can be rewritten as follows. Equation (33) is

$$\Delta_{t+1} = z_{t+1}\Delta_t. \tag{52}$$

Combining (22) and (26), we obtain

$$z_{t+1} = \frac{s\left(\chi^1 + \phi\chi^2\right)}{\gamma(1-s)} \frac{\overline{\delta}_t \omega\left(\overline{\delta}_t, z_t\right)}{\widetilde{\tau}\left(\overline{\delta}_t, z_t\right)}. \tag{53}$$

Finally, from (31) we obtain

$$\Sigma_{t+1} \equiv (\beta+\gamma)\chi^1 - \frac{\alpha\phi\left[\chi^2 - \xi\pi^2\left(\overline{\delta}_{t+1}\right)\right]}{z_{t+1}} + \gamma\left[z_t\eta^1\left(\overline{\delta}_t\right) + \phi\eta^2\left(\overline{\delta}_t\right)\right] - \Delta_t = 0, \tag{54}$$

which implicitly defines

$$\overline{\delta}_{t+1} = \delta\left(\overline{\delta}_t, z_t, z_{t+1}, \Delta_t\right). \tag{55}$$

Equations (52), (53) and (55) form a system of three dynamic equations governing the time path of $\overline{\delta}_t$, Δ_t and z_t. To obtain the elements of the Jacobian matrix associated to this system of equations, we must first obtain the following partial derivatives from the definitions of ω_t and τ_t :[1]

$$\begin{aligned}
\frac{\partial \omega_t}{\partial z_t} &= \frac{\omega^{1-\gamma}}{\gamma}\frac{\xi\phi\overline{\delta}^{v_1-v_2}}{\left(\phi+\overline{\delta}^{v_1-v_2}z\right)^2}, \\
\frac{\partial \omega_t}{\partial \overline{\delta}_t} &= \frac{(v_1-v_2)\omega^{1-\gamma}}{\gamma}\frac{\phi\xi\overline{\delta}^{v_1-v_2-1}z}{\left(\phi+\overline{\delta}^{v_1-v_2}z\right)^2}, \\
\frac{\partial \widetilde{\tau}_t}{\partial z_t} &= \tau_1 \equiv \int_{\overline{\delta}}^{\delta^{\max}} \delta^{1+v_1} f(\delta)\, d\delta, \\
\frac{\partial \widetilde{\tau}_t}{\partial \overline{\delta}_t} &= -f\left(\overline{\delta}\right)\overline{\delta}\left(z\overline{\delta}^{v_1}+\phi(1-\xi)\,\overline{\delta}^{v_2}\right).
\end{aligned}$$

Using (52), we obtain $\partial\Delta_{t+1}/\partial\delta_t = \Delta\left(\partial z_{t+1}/\partial\delta_t\right)$, $\partial\Delta_{t+1}/\partial z_t = \Delta\left(\partial z_{t+1}/\partial z_t\right)$, $\partial\Delta_{t+1}/\partial\Delta_t = z$. Using (53), we obtain

$$\begin{aligned}
\frac{\partial z_{t+1}}{\partial z_t} &= \frac{z}{\omega}\frac{\omega^{1-\gamma}}{\gamma}\frac{\phi\xi\overline{\delta}^{v_1-v_2}}{\left(\phi+\overline{\delta}^{v_1-v_2}z\right)^2} - \frac{z}{\tau}\tau_1, \\
\frac{\partial z_{t+1}}{\partial \overline{\delta}_t} &= \frac{z}{\overline{\delta}} + \frac{z}{\omega}\frac{\omega^{1-\gamma}}{\gamma}\frac{\phi\xi\overline{\delta}^{v_1-v_2-1}z\,(v_1-v_2)}{\left(\phi+\overline{\delta}^{v_1-v_2}z\right)^2} + \frac{z}{\tau}f\left(\overline{\delta}\right)\overline{\delta}\left(z\overline{\delta}^{v_1}+\phi(1-\xi)\,\overline{\delta}^{v_2}\right).
\end{aligned}$$

From (54), we obtain $\partial\Sigma_{t+1}/\partial\overline{\delta}_t = \gamma f\left(\overline{\delta}\right)\left(z\overline{\delta}^{v_1}+\phi\overline{\delta}^{v2}\right)$, $\partial\Sigma_{t+1}/\partial z_t = \gamma\eta^1$, $\partial\Sigma_{t+1}/\partial z_{t+1} = \alpha\left(\phi\chi^2-\xi\phi\pi^2\right)/z^2$, $\partial\Sigma_{t+1}/\partial\Delta_t = -1$, and $\partial\Sigma_{t+1}/\partial\overline{\delta}_{t+1} = -\alpha\phi\xi f\left(\overline{\delta}\right)\overline{\delta}^{v2}/z$. Finally, using (55), we obtain

$$\begin{aligned}
\frac{\partial \overline{\delta}_{t+1}}{\partial z_t} &= \frac{\gamma\eta^1 + \frac{\alpha}{z^2}\left(\phi\chi^2-\phi\xi\pi^2\right)\frac{\partial z_{t+1}}{\partial z_t}}{\frac{\alpha\phi\xi}{z}f\left(\overline{\delta}\right)\overline{\delta}^{v2}}, \\
\frac{\partial \overline{\delta}_{t+1}}{\partial \overline{\delta}_t} &= \frac{\gamma f\left(\overline{\delta}\right)\left(z\overline{\delta}^{v_1}+\phi\overline{\delta}^{v2}\right) + \frac{\alpha}{z^2}\left(\phi\chi^2-\xi\phi\pi^2\right)\frac{\partial z_{t+1}}{\partial \overline{\delta}_t}}{\frac{\alpha\phi\xi}{z}f\left(\overline{\delta}\right)\overline{\delta}^{v2}}, \\
\frac{\partial \overline{\delta}_{t+1}}{\partial \Delta_t} &= -\frac{1}{\frac{\alpha\phi\xi}{z}f\left(\overline{\delta}\right)\overline{\delta}^{v2}}.
\end{aligned}$$

The characteristic polynomial associated to the Jacobian matrix is

$$Q(\lambda) = \left(\frac{\partial\Delta_{t+1}}{\partial\Delta_t}-\lambda\right)P(\lambda) + \Delta\frac{\partial z_{t+1}}{\partial\overline{\delta}_t}\frac{\partial\overline{\delta}_{t+1}}{\partial\Delta_t}\lambda,$$

where $P(\lambda) = \lambda^2 - T\lambda + D$,

$$T = \frac{\partial z_{t+1}}{\partial z_t} + \frac{\partial\overline{\delta}_{t+1}}{\partial\overline{\delta}_t} = \frac{\partial z_{t+1}}{\partial z_t} + \frac{\gamma f\left(\overline{\delta}\right)\left(z\overline{\delta}^{v_1}+\phi\overline{\delta}^{v2}\right) + \frac{\alpha}{z^2}\left(\phi\chi^2-\phi\xi\pi^2\right)\frac{\partial z_{t+1}}{\partial\overline{\delta}_t}}{\frac{\alpha\phi\xi}{z}f\left(\overline{\delta}\right)\overline{\delta}^{v2}},$$

[1] Time subindexes are suppressed to indicate that the derivatives are evaluated at the steady state.

and

$$D = \frac{\partial z_{t+1}}{\partial z_t}\frac{\partial \overline{\delta}_{t+1}}{\partial \overline{\delta}_t} - \frac{\partial z_{t+1}}{\partial \overline{\delta}_t}\frac{\partial \overline{\delta}_{t+1}}{\partial z_t}$$
$$= \frac{\partial z_{t+1}}{\partial z_t}\frac{\gamma f\left(\overline{\delta}\right)\left(z\overline{\delta}^{v_1} + \phi\overline{\delta}^{v2}\right)}{\frac{\alpha\phi\xi}{z} f\left(\overline{\delta}\right)\overline{\delta}^{v2}} - \frac{\partial z_{t+1}}{\partial \overline{\delta}_t}\frac{\gamma\eta^1}{\frac{\alpha\phi\xi}{z} f\left(\overline{\delta}\right)\overline{\delta}^{v2}}.$$

In the bubbleless steady state, $\Delta = 0$. Therefore, $\lambda_1 = z^o < 1$, and λ_2 and λ_3 are the solutions of $P(\lambda) = 0$. To obtain these roots we obtain that

$$D = \frac{\gamma z}{\frac{\alpha\phi\xi}{z} f\left(\overline{\delta}\right)\overline{\delta}^{v2}}\left\{\begin{array}{c}\left(\frac{\omega^{1-\gamma}}{\omega\gamma}\frac{\phi\xi\overline{\delta}^{v_1-v_2}}{\left(\phi+\overline{\delta}^{v_1-v_2}z\right)^2} - \frac{\tau_1}{\tau}\right) f\left(\overline{\delta}\right)\left(z\overline{\delta}^{v_1} + \phi\overline{\delta}^{v2}\right) - \\ \left(\frac{1}{\overline{\delta}} + \frac{\omega^{1-\gamma}\phi\xi\overline{\delta}^{v_1-v_2-1}z(v_1-v_2)}{\omega\gamma\left(\phi+\overline{\delta}^{v_1-v_2}z\right)^2} + \frac{f\left(\overline{\delta}\right)\overline{\delta}\left(z\overline{\delta}^{v_1}+\phi(1-\xi)\overline{\delta}^{v_2}\right)}{\tau}\right)\eta^1\end{array}\right\}.$$

By assuming that $\xi \to 0$, we obtain

$$D = \frac{\gamma z}{\frac{\alpha\phi\xi}{z} f\left(\overline{\delta}\right)\overline{\delta}^{v2}}\left\{-\frac{\tau_1 f\left(\overline{\delta}\right)\left(z\overline{\delta}^{v_1}+\phi\overline{\delta}^{v2}\right)}{\tau} - \left[\frac{1}{\overline{\delta}} + \frac{f\left(\overline{\delta}\right)\overline{\delta}\left(z\overline{\delta}^{v_1}+\phi\overline{\delta}^{v_2}\right)}{\tau}\right]\eta^1\right\} < 0.$$

Therefore, the determinant is negative for values of ξ sufficiently small. Since the determinant is negative, one root is positive and the other one is negative. Next, we get

$$P(1) = 1 - T + D = \frac{1}{\frac{\alpha\phi\xi}{z} f\left(\overline{\delta}\right)\overline{\delta}^{v2}} \times$$
$$\left\{\begin{array}{c}\left(1 - \frac{z}{\omega}\frac{\omega^{1-\gamma}}{\gamma}\frac{\phi\xi\overline{\delta}^{v_1-v_2}}{\left(\phi+\overline{\delta}^{v_1-v_2}z\right)^2} + \frac{z}{\tau}\tau_1\right)\frac{\alpha\phi\xi}{z} f\left(\overline{\delta}\right)\overline{\delta}^{v2} \\ -\gamma f\left(\overline{\delta}\right)\left(z\overline{\delta}^{v_1} + \phi\overline{\delta}^{v2}\right) - \frac{\alpha}{z^2}\left(\phi\chi^2 - \phi\xi\pi^2\right)\frac{\partial z_{t+1}}{\partial \overline{\delta}_t} \\ +\gamma z\left(\frac{\omega^{1-\gamma}}{\omega\gamma}\frac{\phi\xi\overline{\delta}^{v_1-v_2}}{\left(\phi+\overline{\delta}^{v_1-v_2}z\right)^2} - \frac{\tau_1}{\tau}\right) f\left(\overline{\delta}\right)\left(z\overline{\delta}^{v_1} + \phi\overline{\delta}^{v2}\right) \\ -\gamma z\eta^1\left(\frac{1}{\overline{\delta}} + \frac{\omega^{1-\gamma}}{\omega\gamma}\frac{\phi\xi\overline{\delta}^{v_1-v_2-1}(v_1-v_2)}{\left(\phi+\overline{\delta}^{v_1-v_2}z\right)^2} + \frac{f\left(\overline{\delta}\right)\overline{\delta}\left(z\overline{\delta}^{v_1}+\phi(1-\xi)\overline{\delta}^{v_2}\right)}{\tau}\right)\end{array}\right\}.$$

If $\xi \to 0$, we obtain that

$$P(1) = -\frac{1}{\frac{\alpha\phi\xi}{z} f\left(\overline{\delta}\right)\overline{\delta}^{v2}}\left\{\begin{array}{c}\gamma f\left(\overline{\delta}\right)\left(z\overline{\delta}^{v_1} + \phi\overline{\delta}^{v2}\right)\left(1 + \frac{z}{\tau}\tau_1\right) \\ +\left[\frac{z}{\overline{\delta}} + \frac{z}{\tau} f\left(\overline{\delta}\right)\overline{\delta}\left(z\overline{\delta}^{v_1} + \phi\overline{\delta}^{v_2}\right)\right]\left(\gamma\eta^1 + \frac{\alpha}{z^2}\phi\chi^2\right)\end{array}\right\} < 0,$$

which is negative. This implies that $\lambda_2 > 1$ when ξ is sufficiently small. Next, we obtain

$$P(-1) = 1 + T + D = \frac{1}{\frac{\alpha\phi\xi}{z} f\left(\bar{\delta}\right) \bar{\delta}^{v2}} \times$$

$$\times \left\{ \begin{array}{c} \left(1 + \frac{z}{\omega} \frac{\omega^{1-\gamma}}{\gamma} \frac{\phi\xi\bar{\delta}^{v_1 - v_2}}{\left(\phi + \bar{\delta}^{v_1 - v_2} z\right)^2} - \frac{z}{\tau}\tau_1\right) \frac{\alpha\phi\xi}{z} f\left(\bar{\delta}\right) \bar{\delta}^{v2} \\ + \gamma f\left(\bar{\delta}\right)\left(z\bar{\delta}^{v_1} + \phi\bar{\delta}^{v2}\right) \\ + \frac{\alpha\left(\phi\chi^2 - \phi\xi\pi^2\right)\left(\frac{z}{\bar{\bar{\delta}}} + \frac{z\omega^{1-\gamma}}{\omega\gamma} \frac{\phi\xi\bar{\delta}^{v_1 - v_2 - 1} z\left(v_1 - v_2\right)}{\left(\phi + \bar{\delta}^{v_1 - v_2} z\right)^2} + \frac{z f\left(\bar{\delta}\right)\bar{\delta}\left(z\bar{\delta}^{v_1} + \phi(1-\xi)\bar{\delta}^{v_2}\right)}{\tau}\right)}{z^2} \\ + \gamma z\left(\frac{\omega^{1-\gamma}}{\omega\gamma} \frac{\phi\xi\bar{\delta}^{v_1 - v_2}}{\left(\phi + \bar{\delta}^{v_1 - v_2} z\right)^2} - \frac{\tau_1}{\tau}\right) f\left(\bar{\delta}\right)\left(z\bar{\delta}^{v_1} + \phi\bar{\delta}^{v2}\right) \\ - \gamma z\left(\frac{1}{\bar{\bar{\delta}}} + \frac{\omega^{1-\gamma}}{\omega\gamma} \frac{\phi\xi\bar{\delta}^{v_1 - v_2 - 1}\left(v_1 - v_2\right)}{\left(\phi + \bar{\delta}^{v_1 - v_2} z\right)^2} + \frac{f\left(\bar{\delta}\right)\bar{\delta}\left(z\bar{\delta}^{v_1} + \phi(1-\xi)\bar{\delta}^{v_2}\right)}{\tau}\right)\eta^1 \end{array} \right\}.$$

If $\xi \to 0$, we obtain that

$$P(-1) = \frac{1}{\frac{\alpha\phi\xi}{z} f\left(\bar{\delta}\right) \bar{\delta}^{v2}} \left\{ \begin{array}{c} \left(1 - z\frac{\tau_1}{\tau}\right)\gamma f\left(\bar{\delta}\right)\left(z\bar{\delta}^{v_1} + \phi\bar{\delta}^{v2}\right) \\ + \left(\frac{\phi\chi^2\alpha}{z^2} - \gamma\eta^1\right)\left(\frac{z}{\bar{\bar{\delta}}} + \frac{z f\left(\bar{\delta}\right)\bar{\delta}\left(z\bar{\delta}^{v_1} + \phi\bar{\delta}^{v_2}\right)}{\tau}\right) \end{array} \right\},$$

which, using $\Delta = 0$, can be rewritten as

$$P(-1) = \frac{1}{\frac{\alpha\phi\xi}{z} f\left(\bar{\delta}\right) \bar{\delta}^{v2}} \left\{ \begin{array}{c} \left(1 - \frac{z}{\tau}\tau_1\right)\gamma f\left(\bar{\delta}\right)\left(z\bar{\delta}^{v_1} + \phi\bar{\delta}^{v2}\right) \\ + \left(\frac{z}{\bar{\bar{\delta}}} + \frac{z f\left(\bar{\delta}\right)\bar{\delta}\left(z\bar{\delta}^{v_1} + \phi\bar{\delta}^{v_2}\right)}{\tau}\right)\frac{(\beta+\gamma)\chi^1 + \gamma\phi\eta^2}{z} \end{array} \right\} > 0.$$

Therefore, $P(-1) > 0$ and $\lambda_3 > -1$ for ξ sufficiently small. Therefore, for ξ sufficiently small, the three roots are real numbers that satisfy $\lambda_1 \in (0, 1)$, $\lambda_2 > 1$ and $\lambda_3 \in (-1, 0)$. Since z_t is the only state variable, the bubbleless steady state is locally stable.

In the bubbly equilibrium, $z = 1$ and the three roots will be the solution of

$$Q(\lambda) = (1 - \lambda)\left(\lambda^2 - \lambda T + D\right) + \Delta\frac{\partial z_{t+1}}{\partial\bar{\delta}_t}\frac{\partial\bar{\delta}_{t+1}}{\partial\Delta_t}\lambda.$$

To determine the modulus of the roots, we obtain:

$$Q(0) = D = \frac{\gamma}{\alpha\phi\xi f\left(\bar{\delta}\right)\bar{\delta}^{v2}} \left\{ \begin{array}{c} \left(\frac{1}{\omega}\frac{\omega^{1-\gamma}}{\gamma}\frac{\phi\xi\bar{\delta}^{v_1 - v_2}}{\left(\phi + \bar{\delta}^{v_1 - v_2}\right)^2} - \frac{\tau_1}{\tau}\right) f\left(\bar{\delta}\right)\left(\bar{\delta}^{v_1} + \phi\bar{\delta}^{v2}\right) \\ - \left(\begin{array}{c} \frac{1}{\bar{\bar{\delta}}} + \frac{1}{\omega}\frac{\omega^{1-\gamma}}{\gamma}\frac{\phi\xi\bar{\delta}^{v_1 - v_2 - 1}\left(v_1 - v_2\right)}{\left(\phi + \bar{\delta}^{v_1 - v_2}\right)^2} \\ + \frac{1}{\tau} f\left(\bar{\delta}\right)\bar{\delta}\left(\bar{\delta}^{v_1} + \phi(1-\xi)\bar{\delta}^{v_2}\right) \end{array} \right)\eta^1 \end{array} \right\}.$$

If $\xi \to 0$, $Q(0)$ rewrites as

$$Q(0) = \frac{\gamma}{\alpha\phi\xi f\left(\overline{\delta}\right)\overline{\delta}^{v2}} \left\{ \begin{array}{c} -\frac{\tau_1 f\left(\overline{\delta}\right)}{\tau}\left(\overline{\delta}^{v_1} + \phi\overline{\delta}^{v2}\right) \\ -\left(\frac{1}{\overline{\delta}} + \frac{f\left(\overline{\delta}\right)\overline{\delta}\left(\overline{\delta}^{v_1}+\phi\overline{\delta}^{v_2}\right)}{\tau}\right)\eta^1 \end{array} \right\} < 0.$$

We also obtain

$$Q(1) = \Delta\frac{\partial z_{t+1}}{\partial\overline{\delta}_t}\frac{\partial\overline{\delta}_{t+1}}{\partial\Delta_t} = -\frac{\Delta\left(\frac{1}{\overline{\delta}}+\frac{\omega^{-\gamma}}{\gamma}\frac{\phi\xi\overline{\delta}^{v_1-v_2-1}(v_1-v_2)}{\left(\phi+\overline{\delta}^{v_1-v_2}\right)^2}+\frac{1}{\tau}f\left(\overline{\delta}\right)\overline{\delta}\left(\overline{\delta}^{v_1}+\phi\overline{\delta}^{v_2}\right)\right)}{\alpha\xi\phi f\left(\overline{\delta}\right)\overline{\delta}^{v2}} < 0.$$

Finally,

$$Q(-1) = 2(1+T+D) - \Delta\frac{\partial z_{t+1}}{\partial\overline{\delta}_t}\frac{\partial\overline{\delta}_{t+1}}{\partial\Delta_t} = 2P(-1) - Q(1) > 0,$$

which is positive when ξ is small since then $P(-1) > 0$ and $Q(1) < 0$.

Since $Q(0) < 0$, $Q(1) < 0$, $Q(-1) > 0$, $Q(\infty) = -\infty$ and $Q(-\infty) = \infty$, it follows that $\lambda_1 \in (-1, 0)$. However, the other two roots can be real or complex numbers with a modulus that can be either larger or smaller than one. It follows that the bubbly steady state can be either locally stable or saddle path stable.

E Total factor productivity

In this appendix, we study the effect of $\overline{\delta}_t$ and z_t on TFP. To this end, we rewrite (38) as

$$TFP_t = A\frac{z_t\int_{\overline{\delta}_t}^{\delta_{\max}}\delta^i g^1\left(\delta^i\right)d\delta^i + (1-\xi)\phi\int_{\overline{\delta}_t}^{\delta_{\max}}\delta^i g^2\left(\delta^i\right)d\delta^i}{z_t\int_{\overline{\delta}_t}^{\delta_{\max}}g^1\left(\delta^i\right)d\delta^i + (1-\xi)\phi\int_{\overline{\delta}_t}^{\delta_{\max}}g^2\left(\delta^i\right)d\delta^i},$$

where

$$g^j\left(\delta^i\right) = \frac{\left(\delta^i\right)^{v_j}f\left(\delta^i\right)}{1-\theta sA\delta^i/R_{t+1}} \text{ for all } j = 1, 2.$$

We first obtain

$$\begin{aligned}\frac{\partial TFP_t}{\partial\overline{\delta}_t} &= -\frac{A}{\left(z\int_{\overline{\delta}_t}^{\delta_{\max}}g^1\left(\delta^i\right)d\delta^i + (1-\xi)\phi\int_{\overline{\delta}_t}^{\delta_{\max}}g^2\left(\delta^i\right)d\delta^i\right)^2} \\ &\times\left\{\begin{array}{c}\left[z\overline{\delta}g^1\left(\overline{\delta}_t\right) + (1-\xi)\phi\overline{\delta}g^2\left(\overline{\delta}_t\right)\right]\times \\ \times\left[z\int_{\overline{\delta}_t}^{\delta_{\max}}g^1\left(\delta^i\right)d\delta^i + \phi(1-\xi)\int_{\overline{\delta}_t}^{\delta_{\max}}g^2\left(\delta^i\right)d\delta^i\right] \\ -\left[zg^1\left(\overline{\delta}_t\right) + (1-\xi)\phi g^2\left(\overline{\delta}_t\right)\right] \\ \times\left[z\int_{\overline{\delta}_t}^{\delta_{\max}}\delta^i g^1\left(\delta^i\right)d\delta^i + \phi(1-\xi)\int_{\overline{\delta}_t}^{\delta_{\max}}\delta^i g^2\left(\delta^i\right)d\delta^i\right]\end{array}\right\},\end{aligned}$$

which can be rewritten as

$$\frac{\partial TFP_t}{\partial \overline{\delta}_{t+1}} = -\frac{A\left[zg^1\left(\overline{\delta}_t\right) + (1-\xi)\,\phi g^2\left(\overline{\delta}_t\right)\right]}{\left(z\int_{\overline{\delta}_t}^{\delta_{\max}} g^1\left(\delta^i\right)d\delta^i + (1-\xi)\,\phi\int_{\overline{\delta}_t}^{\delta_{\max}} g^2\left(\delta^i\right)d\delta^i\right)^2}$$

$$\times\left\{\begin{array}{c}\left[z\int_{\overline{\delta}_t}^{\delta_{\max}}\overline{\delta}_t g^1\left(\delta^i\right)d\delta^i + \phi(1-\xi)\int_{\overline{\delta}_t}^{\delta_{\max}}\overline{\delta}_t g^2\left(\delta^i\right)d\delta^i\right]\\ -\left[z\int_{\overline{\delta}_t}^{\delta_{\max}}\delta^i g^1\left(\delta^i\right)d\delta^i + \phi(1-\xi)\int_{\overline{\delta}_t}^{\delta_{\max}}\delta^i g^2\left(\delta^i\right)d\delta^i\right]\end{array}\right\} > 0.$$

Indeed, the last term into the brackets is negative because the values of δ^i considered are larger or equal to $\overline{\delta}$.

We next obtain

$$\frac{\partial TFP_t}{\partial z_t} = A(1-\xi)\,\phi\frac{\int_{\overline{\delta}_t}^{\delta_{\max}}\delta^i g^1\left(\delta^i\right)d\delta^i\int_{\overline{\delta}_t}^{\delta_{\max}} g^2\left(\delta^i\right)d\delta^i - \int_{\overline{\delta}_t}^{\delta_{\max}} g^1\left(\delta^i\right)d\delta^i\int_{\overline{\delta}_t}^{\delta_{\max}}\delta^i g^2\left(\delta^i\right)d\delta^i}{\left(z\int_{\overline{\delta}_t}^{\delta_{\max}} g^1\left(\delta^i\right)d\delta^i + (1-\xi)\,\phi\int_{\overline{\delta}_t}^{\delta_{\max}} g^2\left(\delta^i\right)d\delta^i\right)^2},$$

which is positive if

$$\int_{\overline{\delta}_t}^{\delta_{\max}}\delta^i g^1\left(\delta^i\right)d\delta^i\int_{\overline{\delta}_t}^{\delta_{\max}} g^2\left(\delta^i\right)d\delta^i - \int_{\overline{\delta}_t}^{\delta_{\max}} g^1\left(\delta^i\right)d\delta^i\int_{\overline{\delta}_t}^{\delta_{\max}}\delta^i g^2\left(\delta^i\right)d\delta > 0.$$

After some manipulation, we can rewrite the former inequality as

$$\int_{\overline{\delta}_t}^{\delta_{\max}}\int_{\overline{\delta}_t}^{\delta_{\max}} g^1\left(\delta^j\right)g^2\left(\delta^i\right)\left(\delta^j-\delta^i\right)d\delta^i d\delta^j > 0,$$

which is equivalent to

$$\int_{\delta^i}^{\delta_{\max}}\int_{\overline{\delta}_t}^{\delta_{\max}} g^1\left(\delta^j\right)g^2\left(\delta^i\right)\left(\delta^j-\delta^i\right)d\delta^i d\delta^j +$$
$$\int_{\overline{\delta}_t}^{\delta^i}\int_{\overline{\delta}_t}^{\delta_{\max}} g^1\left(\delta^j\right)g^2\left(\delta^i\right)\left(\delta^j-\delta^i\right)d\delta^i d\delta^j > 0. \tag{56}$$

In (56), we split the inequality in two parts. In the first one, $\delta^j > \delta^i$ and in the second $\delta^j < \delta^i$. Using this same separation, we deduce that

$$\int_{\overline{\delta}_t}^{\delta^i}\int_{\overline{\delta}_t}^{\delta_{\max}} g^1\left(\delta^j\right)g^2\left(\delta^i\right)\left(\delta^j-\delta^i\right)d\delta^i d\delta^j \tag{57}$$
$$= -\int_{\delta^i}^{\delta_{\max}}\int_{\overline{\delta}_t}^{\delta_{\max}} g^2\left(\delta^j\right)g^1\left(\delta^i\right)\left(\delta^j-\delta^i\right)d\delta^i d\delta^j$$

We use (57) to rewrite (56) as the following inequality in which $\delta^j > \delta^i$:

$$\int_{\delta^i}^{\delta_{\max}}\int_{\overline{\delta}_t}^{\delta_{\max}} g^1\left(\delta^j\right)g^2\left(\delta^i\right)\left(\delta^j-\delta^i\right)d\delta^i d\delta^j$$
$$-\int_{\delta^i}^{\delta_{\max}}\int_{\overline{\delta}_t}^{\delta_{\max}} g^2\left(\delta^j\right)g^1\left(\delta^i\right)\left(\delta^j-\delta^i\right)d\delta^i d\delta^j > 0.$$

Using the expression of $g^j(\delta)$, we rewrite the former inequality as

$$\int_{\delta^i}^{\delta_{\max}} \int_{\bar{\delta}_t}^{\delta_{\max}} \left(\frac{f(\delta^i)}{1 - \theta s A \delta^i / R_{t+1}} \right)^2 (\delta^i)^{v_1 + v_2} (\delta^j - \delta^i) \left[\left(\frac{\delta^j}{\delta^i} \right)^{v_1} - \left(\frac{\delta^j}{\delta^i} \right)^{v_2} \right] d\delta^i d\delta^j > 0.$$

Since $\delta^j > \delta^i$, we deduce that the inequality holds if and only if $v_1 > v_2$. This proves that $\partial TFP_t / \partial z_t > 0$ if and only if $v_1 > v_2$.